Cascade process of linked quantum vortex loops

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Goals

- Detailed study of the **cascade process** of two linked quantum vortex loops (2 $\rightarrow$ 1-folded $\rightarrow$ 2 $\rightarrow$ 3) under the Gross-Pitaevskii equation (GPE),

$$\frac{\partial \psi}{\partial t} = \frac{\mathrm{i}}{2}\nabla^2\psi + \frac{\mathrm{i}}{2}\left(1 - |\psi|^2\right)\psi, \qquad |\psi| \rightarrow 1 \text{ as } |x| \rightarrow \infty\,.$$

- **Accurate numerical calculation** of geometric and topological properties (Wr, Tw, Lk, SL, N, T).
- Investigation of **possible helicity conservation** throughout the whole process

$$H = \sum_{i \neq j} \Gamma_i \Gamma_j Lk_{ij} + \sum_i \Gamma_i^2 (Wr_i + Tw_i), \qquad \Gamma_i = 2\pi\,.$$

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Numerics

- 2nd-order **Strang splitting** (time), **Fourier** (space), see Koplik & Levine *Phys. Rev. Lett.* **71**, 1993. Boundary conditions must be **periodic**, computational domain doubled in each direction, “mirror” vortex rings.
- The method **conserves mass exactly**, previously used for simulating vortex reconnection (see Zuccher *et al. Phys. Fluids* **24**, 2012 and Zuccher & Ricca *Phys. Rev. E* **92**, 2015).
- Each vortex tube contributes to the **initial condition** with $\psi = \sqrt{\rho_4} e^{i\theta}$, where $\rho_4(r) = \frac{a_1 r^2 + a_2 r^4 + a_3 r^6 + a_4 r^8}{1 + b_1 r^2 + b_2 r^4 + b_3 r^6 + a_4 r^8}$ (see Caliari & Zuccher arXiv preprint arXiv:1603.05022, 2016).
- $\Delta x = \Delta y = \Delta z = \xi/3$, 150^3 points, $\Delta t = 1/80 = 0.0125$. ***Local* higher resolution** ($\xi/10$) during the post-processing by employing Nonuniform Fast Fourier Transform (see Caliari & Zuccher *Fast evaluation of 3d Fourier series in MATLAB with an application to quantum vortex reconnections*, 2016, in preparation).

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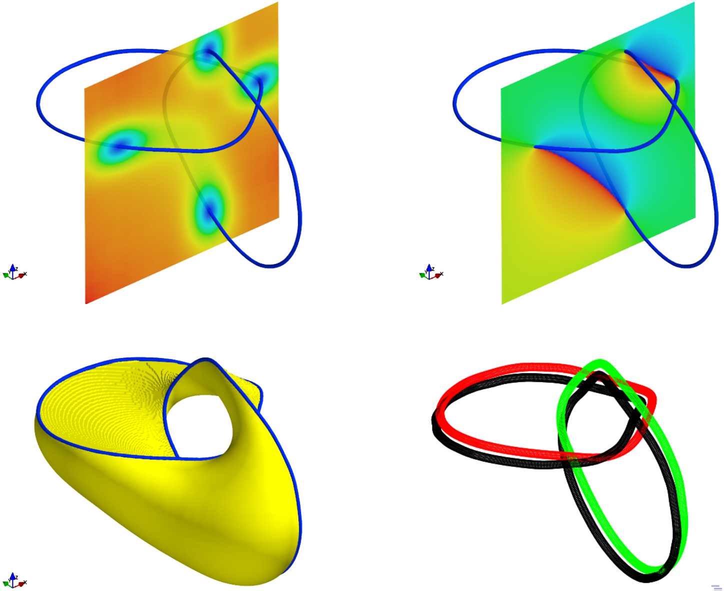
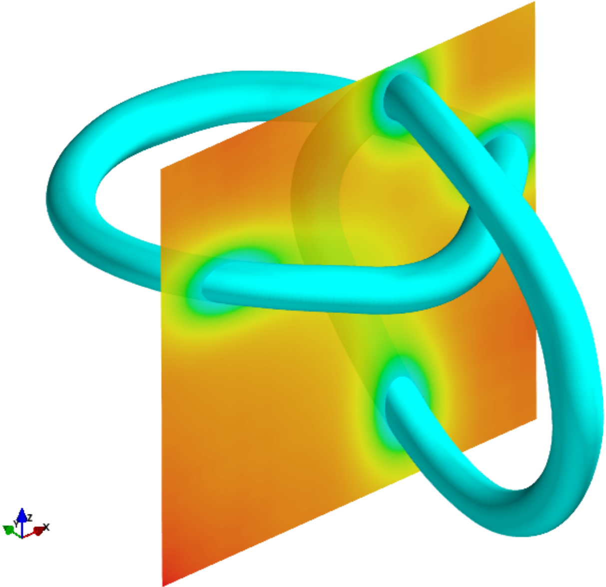
Centerlines and ribbons, $t = 23$



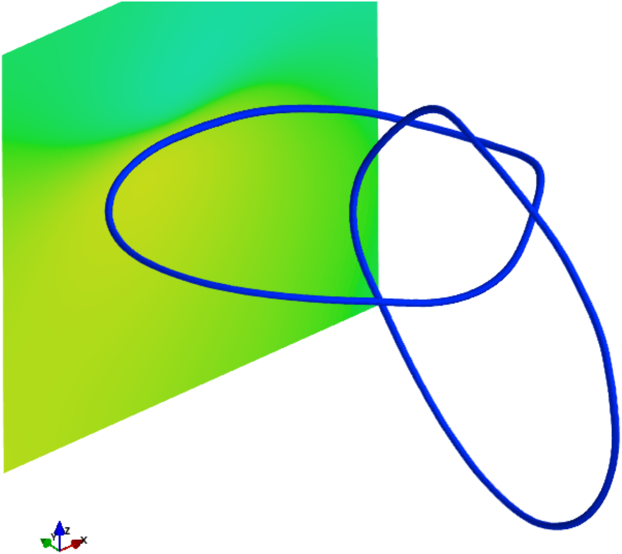
Density $\rho = |\psi|^2$, isosurface

Back



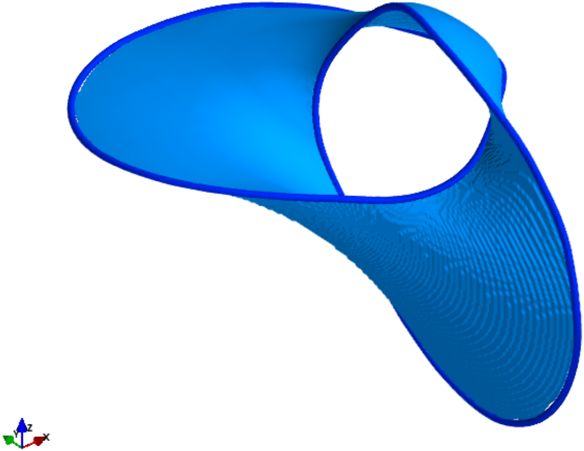
Phase $\theta = \angle\psi$, scalar cut plane

Back



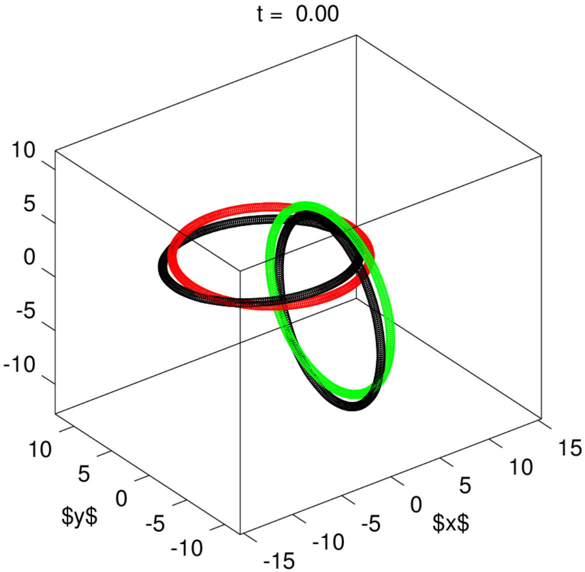
Phase $\theta = \angle\psi$, isosurface

Back

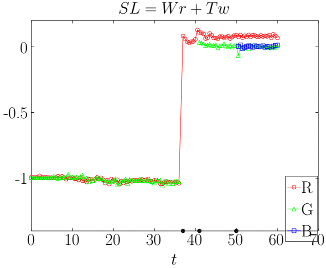


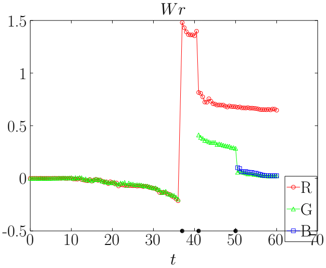
Vortex centerlines and ribbon edges (black)

Next



Geometric quantities

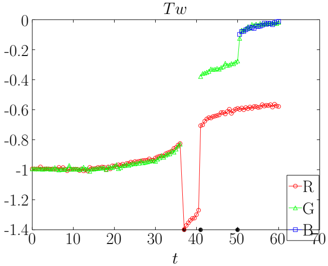




$$SL_i = Wr_i + Tw_i$$

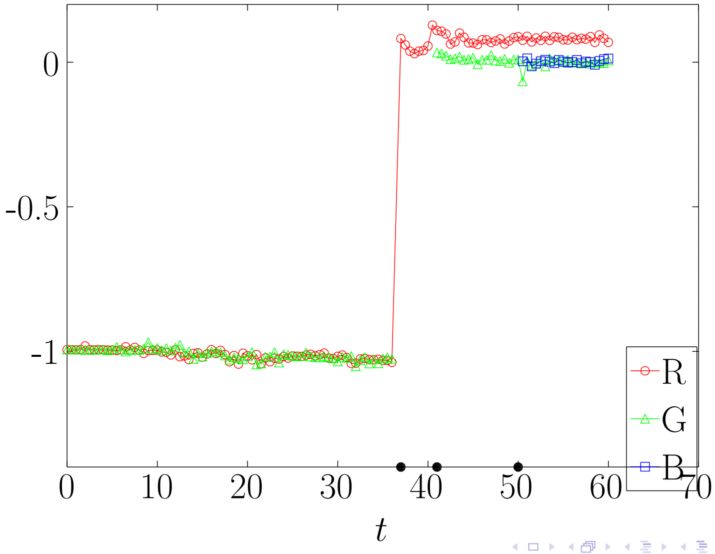
$$Wr_i = \frac{1}{4\pi} \int_{C_i} \int_{C_i} \frac{\mathbf{x} - \mathbf{x}^*}{\|\mathbf{x} - \mathbf{x}^*\|^3} \cdot (\,\mathrm{d}\mathbf{x} \times \mathrm{d}\mathbf{x}^*)$$

$$Tw_i = \frac{1}{2\pi} \int_{C_i} \left(\mathbf{U} \times \frac{\mathrm{d}\mathbf{U}}{\mathrm{d}s} \right) \cdot \hat{\mathbf{n}}\,\mathrm{d}s$$

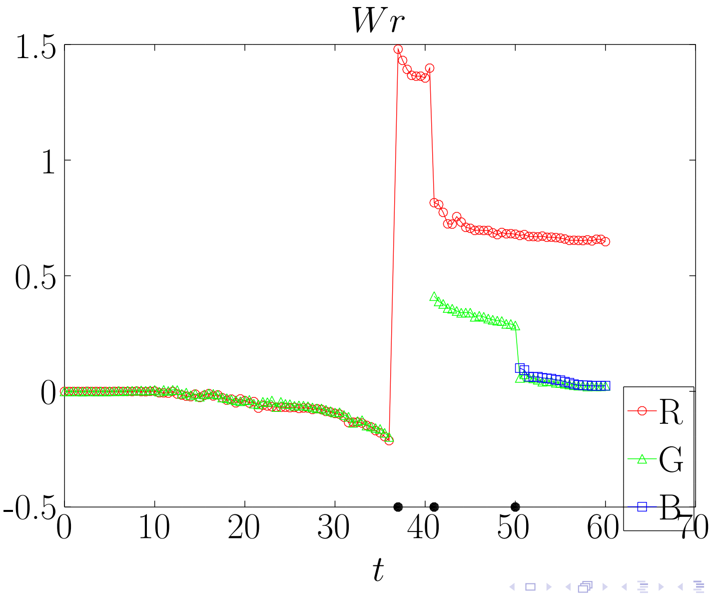


Self-linking number

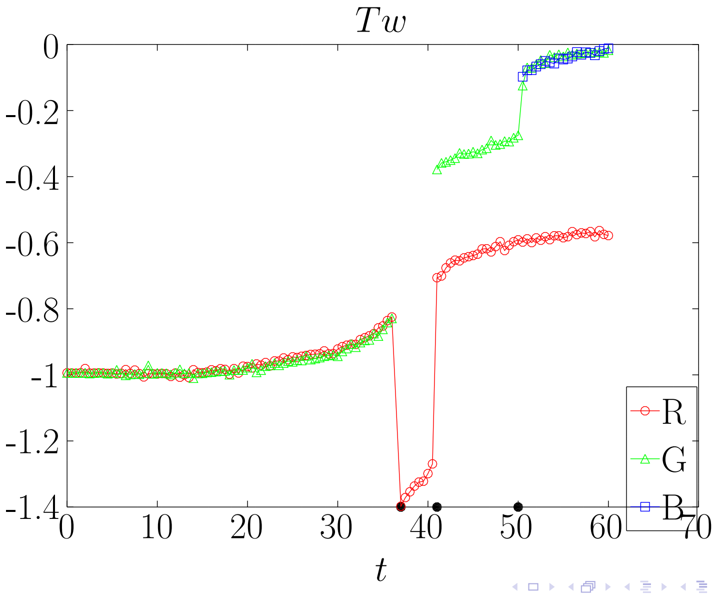
$$SL = Wr + Tw$$



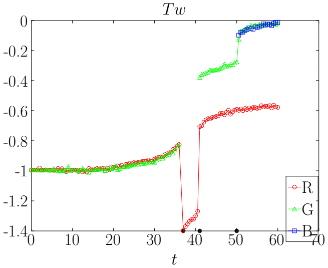
Writhing number

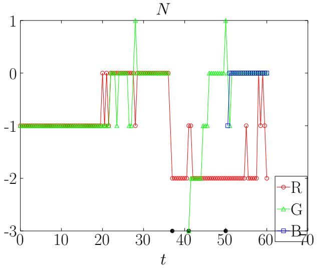


Twist



Total twist

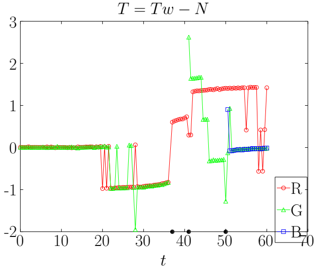




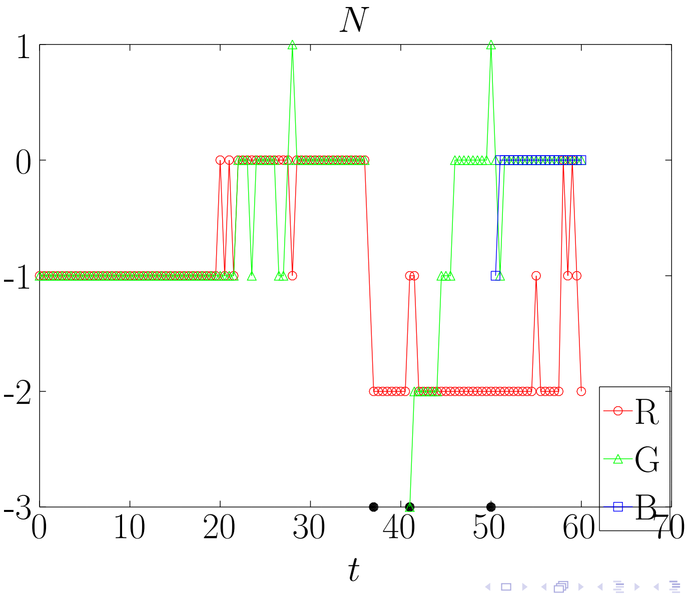
$$Tw_i = N_i + T_i$$

$$N_i = \frac{1}{2\pi}\int_{C_i} \frac{\mathrm{d}\varphi(s)}{\mathrm{d}s}\,\mathrm{d}s = \frac{[\varphi]_{C_i}}{2\pi}$$

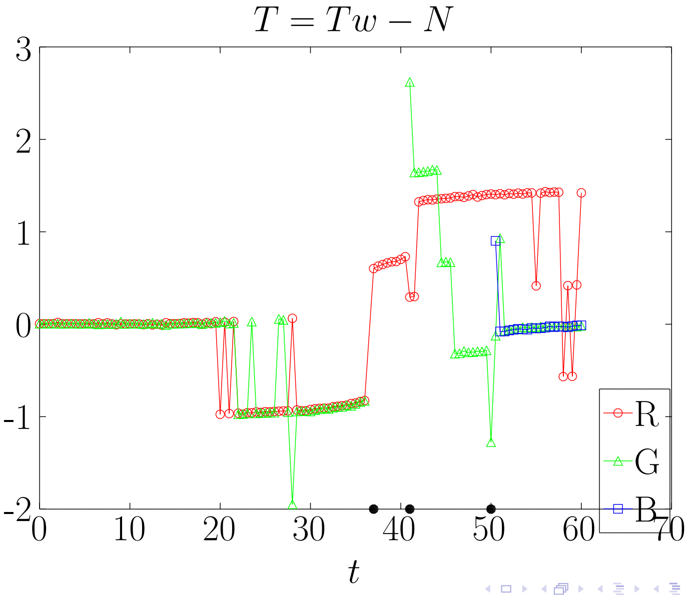
$$T_i = \frac{1}{2\pi}\int_{C_i} \tau(s)\,\mathrm{d}s$$



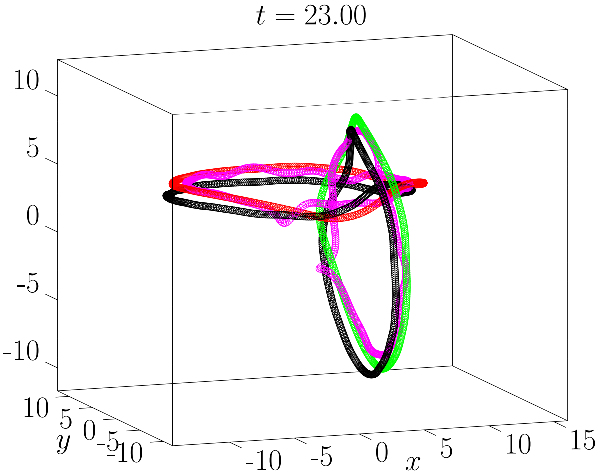
Intrinsic twist



Total torsion



Inflection points



The singularity of torsion at inflection points is integrable, see Moffatt & Ricca *Proc. R. Soc. Lond. A* **439**, 1992.

Conclusions

- Linked vortex rings **evolve towards unlinked vortex loops**. As $t \to \infty$, $Lk_{ij} = 0$, so:

$$H = \sum_i \Gamma_i^2 (Wr_i + Tw_i) \ .$$

- Accurate numerical calculation of all geometric and topological quantities shows that ***Wr* and *Tw* compensate one another**.
- GPE is Hamiltonian and by the Madelung transformation results in compressible Navier-Stokes equations with quantum pressure and quantum stress. **GPE seems to conserve helicity while allowing vortex reconnection**.

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