Optimal perturbations and control of nonlinear boundary layer

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Optimization

Objectives

$\Rightarrow$ Boundary layer, in the black–box fashion, receives boundary conditions and initial conditions as inputs: what is the most dangerous initial condition which maximizes the perturbation energy? **Optimal perturbation**

$\Rightarrow$ For the most dangerous initial condition, what is the best suction to be applied at the wall in order to minimize the perturbation energy? **Optimal control** and **robust control**

$\Rightarrow$ What happens if the initial energy is gradually increased and the **nonlinear** regime reached?

Methodology

$\smile$ Use of an optimization technique based on the solution of the linear **adjoint equations** corresponding to the nonlinear direct ones.

Problem Formulation

- 3D incompressible and steady boundary layer equations in conservative form
- u normalized with respect to U_∞, v and w with respect to $Re^{-1/2}U_\infty$ ($Re = U_\infty L/\nu$)
- Flow field $\mathbf{V} = (u, v, w)$ subdivided in two contributions $\mathbf{V}_0$ (independent of z) and $\bar{\mathbf{v}}$ (dependent on z)

$$\mathbf{V}(x,y,z) = \mathbf{V}_0(x,y) + \bar{\mathbf{v}}(x,y,z)$$

- Kinetic energy of $\bar{\mathbf{v}}(x,y,z)$ taken as a measure of the level of perturbation:

$$E(x) = \int_{-Z}^{Z}\int_0^\infty [|\bar{u}|^2 + Re^{-1}(|\bar{v}|^2 + |\bar{w}|^2)]dy\,dz$$

- Objective function (to be minimized or maximized):

$$\mathcal{J} = \alpha_1 G_{\text{out}} + \alpha_2 G_{\text{mean}}$$

under the hypotheses $Re \to \infty$ and $\bar{u}|_{x=0} = 0$:

$$G_{\text{out}} = \frac{E_{\text{out}}}{E_{\text{in}}} = Re\frac{\int_{-Z}^{Z}\int_0^\infty [|\bar{u}|^2]dy\,dz}{\left[\int_{-Z}^{Z}\int_0^\infty [|\bar{v}|^2 + |\bar{w}|^2]dy\,dz\right]_{x=0}}; \quad G_{\text{mean}} = \frac{E_{\text{mean}}}{E_{\text{in}}} = Re\frac{\int_{-Z}^{Z}\int_0^\infty\int_0^X [|\bar{u}|^2]dx\,dy\,dz}{\left[\int_{-Z}^{Z}\int_0^\infty [|\bar{v}|^2 + |\bar{w}|^2]dy\,dz\right]_{x=0}}$$

Constrained Optimization

Constraints on the initial energy E_{in} and control energy E_{w}

$$E_{\text{in}} = \left[\int_{-Z}^{Z} \int_{0}^{\infty} [|\bar{v}|^2 + |\bar{w}|^2] dy\, dz \right]_{x=0} = E_0; \qquad E_{\text{w}} = \left[\int_{x_{in}}^{X} |v_{\text{w}}|^2 \, dx \right]_{y=0} = E_{\text{w}0}$$

Lagrange multipliers technique. Functional $\mathcal{L}(u, v, w, p, \bar{v}_0, v_{\text{w}}, a, b, c, d, \lambda_0, \lambda_{\text{w}})$:

$$\begin{aligned} \mathcal{L} = \ \mathcal{J} \ & + \int_{-Z}^{Z} \int_{0}^{\infty} \int_{0}^{X} a[u_x + v_y + w_z] dx\, dy\, dz \\ & + \int_{-Z}^{Z} \int_{0}^{\infty} \int_{0}^{X} b[(uu)_x + (uv)_y + (uw)_z - u_{yy} - u_{zz}] dx\, dy\, dz \\ & + \int_{-Z}^{Z} \int_{0}^{\infty} \int_{0}^{X} c[(uv)_x + (vv)_y + (vw)_z + p_y - v_{yy} - v_{zz}] dx\, dy\, dz \\ & + \int_{-Z}^{Z} \int_{0}^{\infty} \int_{0}^{X} d[(uw)_x + (vw)_y + (ww)_z + p_x - w_{yy} - w_{zz}] dx\, dy\, dz \\ & + \lambda_0 [E_{\text{in}}(\bar{v}_0) - E_0] + \lambda_{\text{w}} [E_{\text{w}}(v_{\text{w}}) - E_{\text{w}0}] \end{aligned}$$

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta u}\delta u + \frac{\delta\mathcal{L}}{\delta v}\delta v + \frac{\delta\mathcal{L}}{\delta w}\delta w + \frac{\delta\mathcal{L}}{\delta p}\delta p + \frac{\delta\mathcal{L}}{\delta \bar{v}_0}\delta \bar{v}_0 + \frac{\delta\mathcal{L}}{\delta v_{\text{w}}}\delta v_{\text{w}} + \frac{\delta\mathcal{L}}{\delta a}\delta a + \frac{\delta\mathcal{L}}{\delta b}\delta b + \frac{\delta\mathcal{L}}{\delta c}\delta c + \frac{\delta\mathcal{L}}{\delta d}\delta d + \frac{\delta\mathcal{L}}{\delta \lambda_0}\delta \lambda_0 + \frac{\delta\mathcal{L}}{\delta \lambda_{\text{w}}}\delta \lambda_{\text{w}} = 0$$

$$\frac{\delta\mathcal{L}}{\delta u}\delta u = \lim_{\epsilon \to 0} \frac{\mathcal{L}(u + \epsilon\delta u, v, w, p, \bar{v}_0, v_{\text{w}}, a, b, c, d, \lambda_0, \lambda_{\text{w}}) - \mathcal{L}(u, v, w, p, \bar{v}_0, v_{\text{w}}, a, b, c, d, \lambda_0, \lambda_{\text{w}})}{\epsilon}$$

Adjoint problem

From integration by parts ($a^* = a + 2bu$):

$$\begin{aligned} c_y + d_z &= 0 \\ a^*_x - 2u_x b + b_y v + b_z w + c_z v + d_x v + d_x w + b_{yy} + b_{zz} &= \alpha_2 u \\ a^*_y - 2bu_y - b_y u + c_x u + 2c_y v + d_y w + c_z w + c_{yy} + c_{zz} &= 0 \\ a^*_z - 2bu_z - b_z u + c_z v + d_y v + d_x u + 2d_z w + d_{yy} + d_{zz} &= 0 \end{aligned}$$

with boundary conditions

$$\begin{aligned} b = 0 \quad &\text{at} \quad y = 0 & c = 0 \quad &\text{for} \quad y \to \infty \\ a^* - 2bu + c_y = 0 \quad &\text{at} \quad y = 0 & a^* - ub + c_y = 0 \quad &\text{for} \quad y \to \infty \\ d = 0 \quad &\text{at} \quad y = 0 & d = 0 \quad &\text{for} \quad y \to \infty \end{aligned}$$

and “initial conditions” at $x = X$

$$\begin{aligned} &c = 0 & &\text{at} \quad x = X \\ &d = 0 & &\text{at} \quad x = X \\ &\int_{-Z}^{Z} \int_0^{\infty} a^* dy\, dz + \alpha_1 \frac{\delta G_{\text{out}}}{\delta u} = 0 & &\text{at} \quad x = X \end{aligned}$$

From the integration by parts also “coupling conditions” between the adjoint and direct problem are obtained:

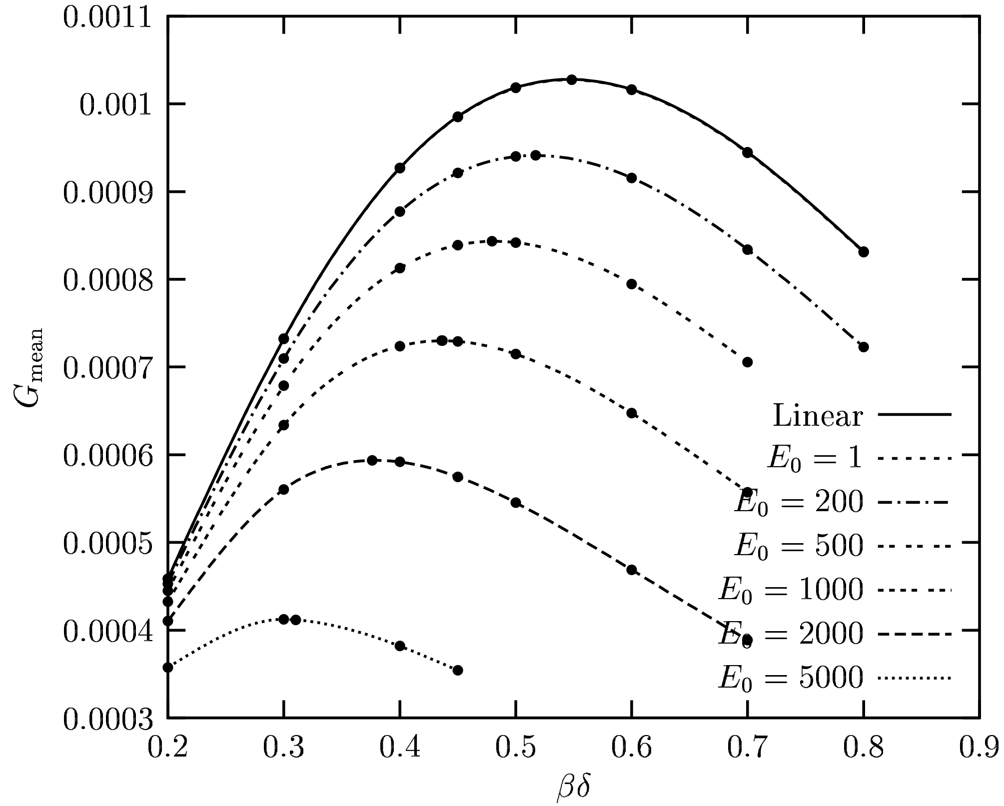
$$\int_{-Z}^{Z} \int_0^{\infty} c\, dy\, dz + \lambda_0 \frac{\delta E_{\text{in}}}{\delta \overline{v}} = 0 \text{ at } x = 0; \qquad \int_0^{X} c\, dx - \lambda_{\text{w}} \frac{\delta E_{\text{w}}}{\delta v_{\text{w}}} = 0 \text{ at } y = 0$$

Iterative optimization procedure

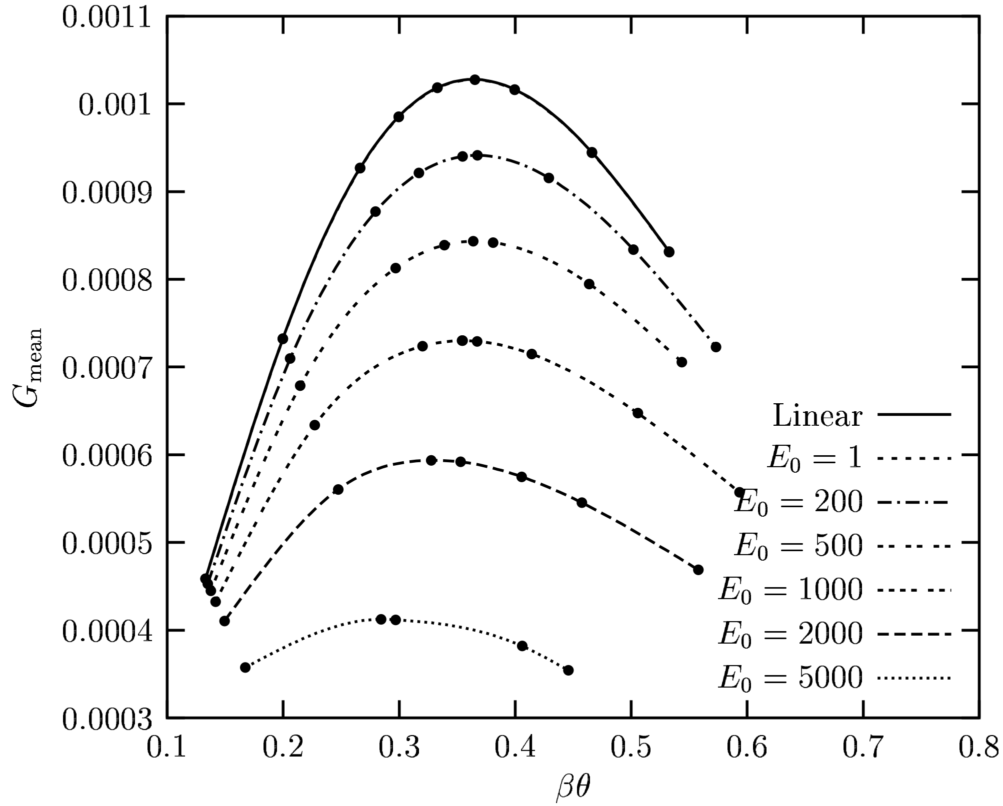
$x = 0$	$0 < x < X$	$x = X$
		$*$
$v_0^{(1)}, \quad v_{\mathrm{w}}^{(1)}$	$u^{(1)}, v^{(1)}, w^{(1)}, p^{(1)}$ $\Longrightarrow$	$\mathcal{J}^{(1)}, \quad \dfrac{\delta G_{\mathrm{out}}}{\delta u}^{(1)}$ $\int_{-Z}^{Z}\int_0^\infty a^{*(1)}\,dy\,dz = -\alpha_1 \dfrac{\delta G_{\mathrm{out}}}{\delta u}^{(1)}$
		$\Downarrow$
$c^{(1)}$ $\int_{-Z}^{Z}\int_0^\infty c^{(1)}\,dy\,dz + \lambda_0 \dfrac{\delta E_{\mathrm{in}}}{\delta v}^{(2)} = 0$ $\int_0^X c^{(1)}\,dx - \lambda_{\mathrm{w}} \dfrac{\delta E_{\mathrm{w}}}{\delta v_{\mathrm{w}}}^{(2)} = 0$	$\Longleftarrow$ $a^{*(1)}, b^{(1)}, c^{(1)}, d^{(1)}$	$a^{*(1)}$ $c^{(1)} = 0$ $d^{(1)} = 0$
$\Downarrow$		
$v_0^{(2)}, \quad v_{\mathrm{w}}^{(2)}$	$u^{(2)}, v^{(2)}, w^{(2)}, p^{(2)}$ $\Longrightarrow$	$\mathcal{J}^{(2)}$ $\Downarrow$ $\left\| \mathcal{J}^{(n)} - \mathcal{J}^{(n-1)} \right\| < \epsilon$ no $\downarrow$ repeat from $*$ / yes $\downarrow$ done

Optimal perturbation

Gain $G_{\text{mean}} = E_{\text{mean}}/E_0$ for different values of initial energy E_0 and wavenumber β



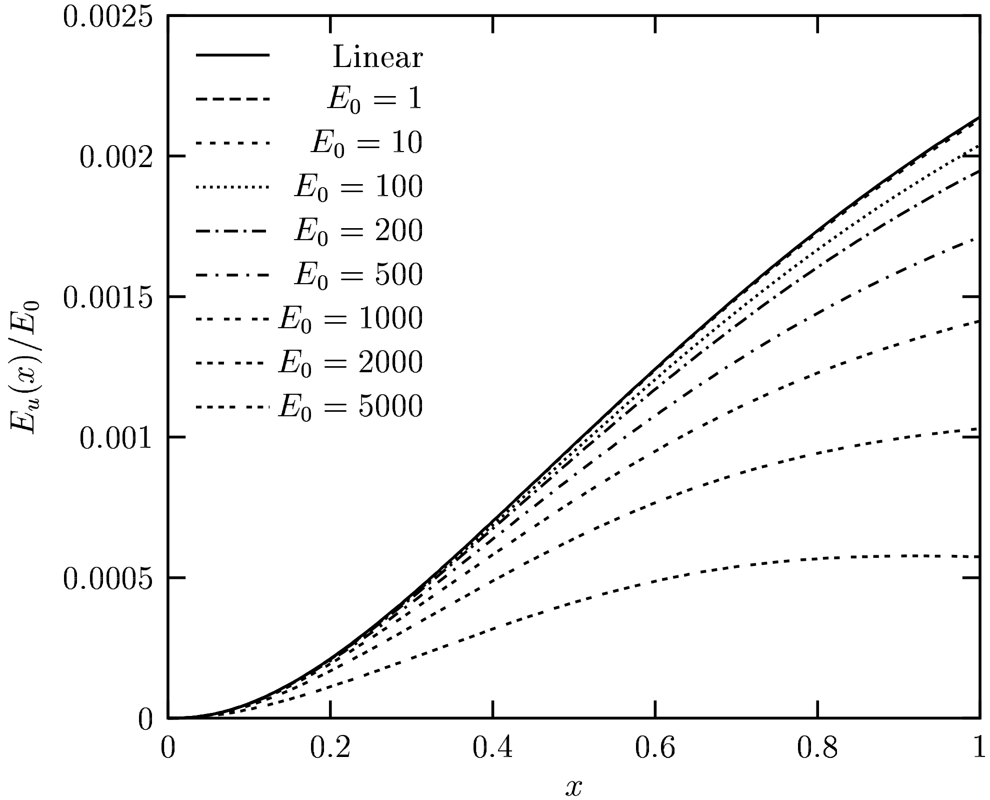
$$\delta = L/\sqrt{Re_L} = \sqrt{\nu L/U_\infty}$$



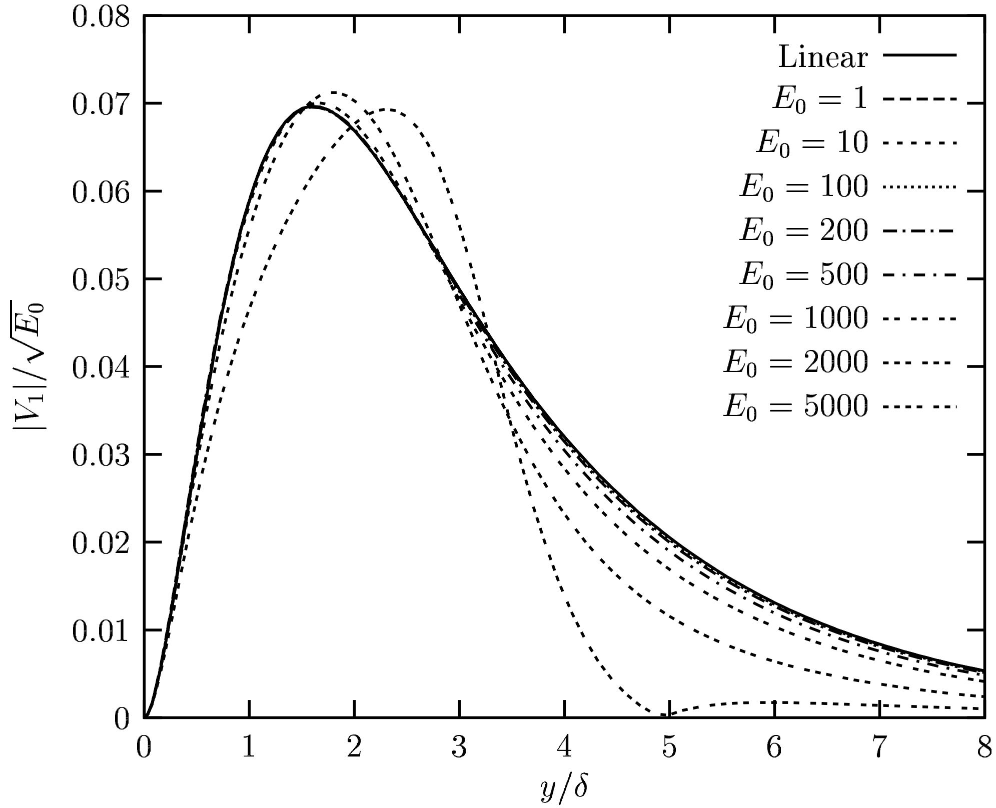
$$\theta = \int_0^{+\infty} \frac{U_0}{U_\infty}\left(1 - \frac{U_0}{U_\infty}\right) dy$$

Optimal perturbation – $\beta\delta = 0.45$

Optimal perturbation for varying E_0 and at $\beta\delta$ fixed



Energy behavior $E_u(x)/E_0$

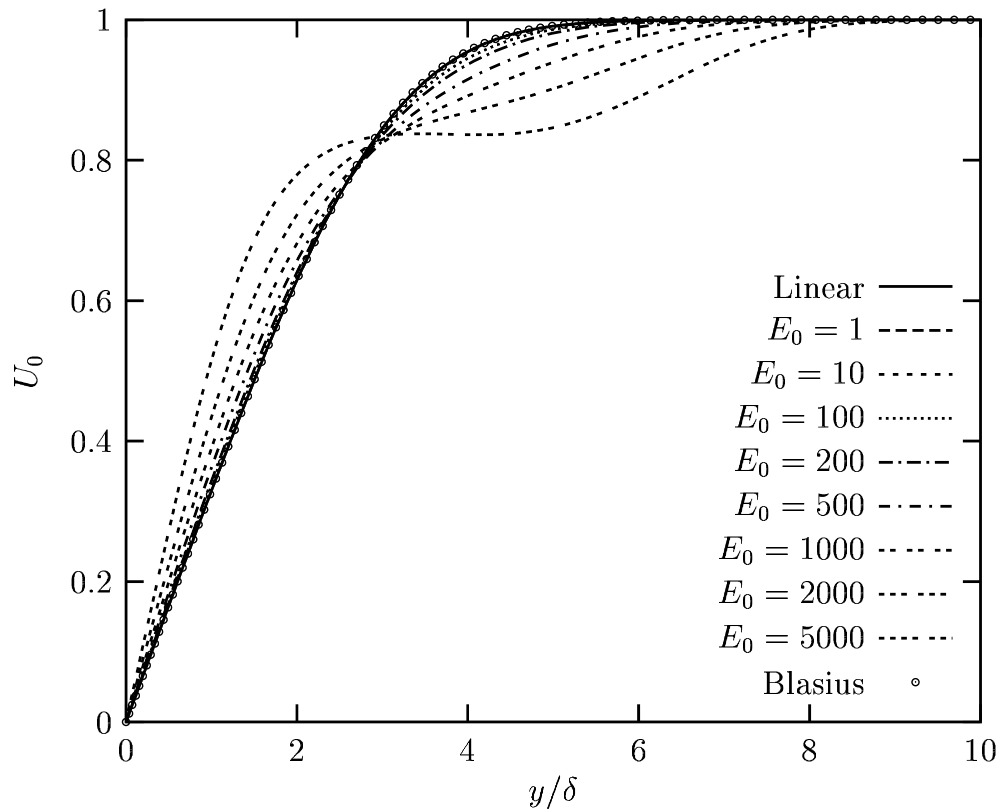


Initial perturbation $|V_1|/\sqrt{E_0}$

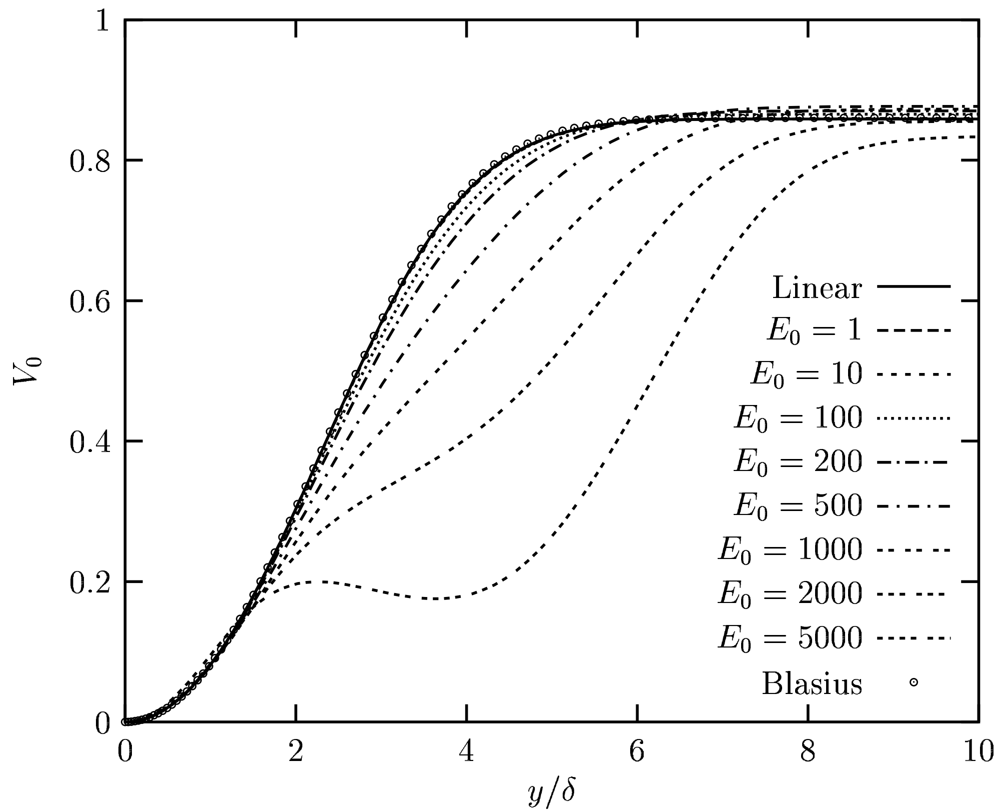
$\Rightarrow$ Saturation for high initial energy?

Optimal perturbation – $\beta\delta = 0.45$

Mean flow contribution (independent of z). Profiles at the final station $x = 1$



U_0 with varying E_0

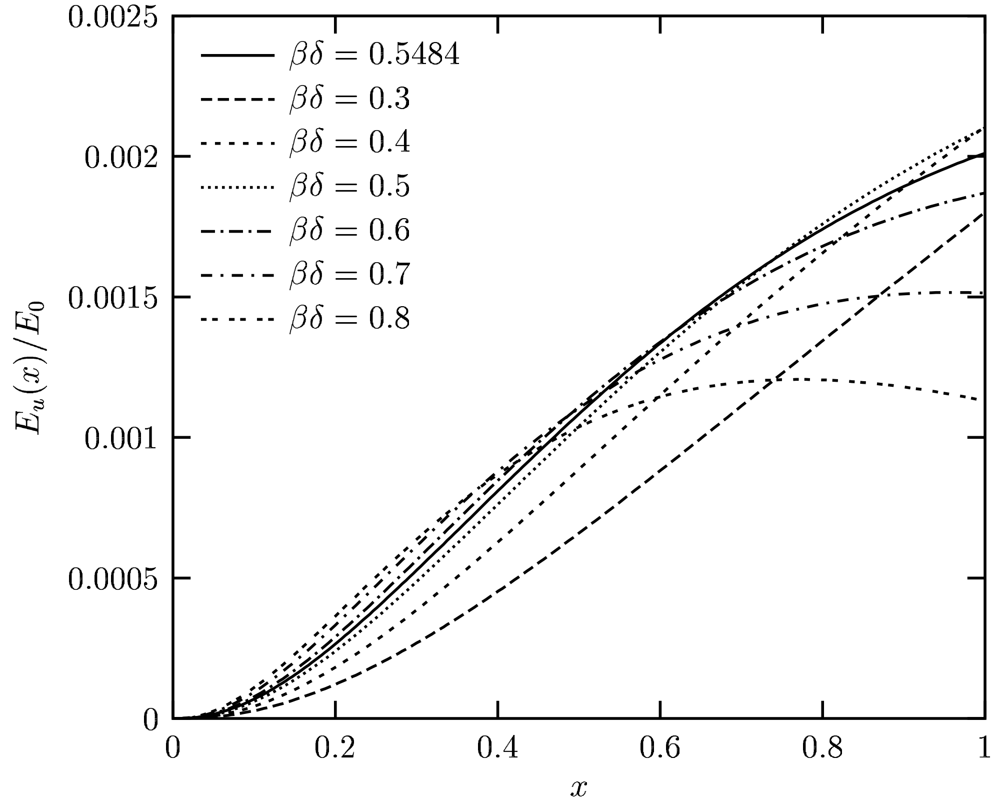


V_0 with varying E_0

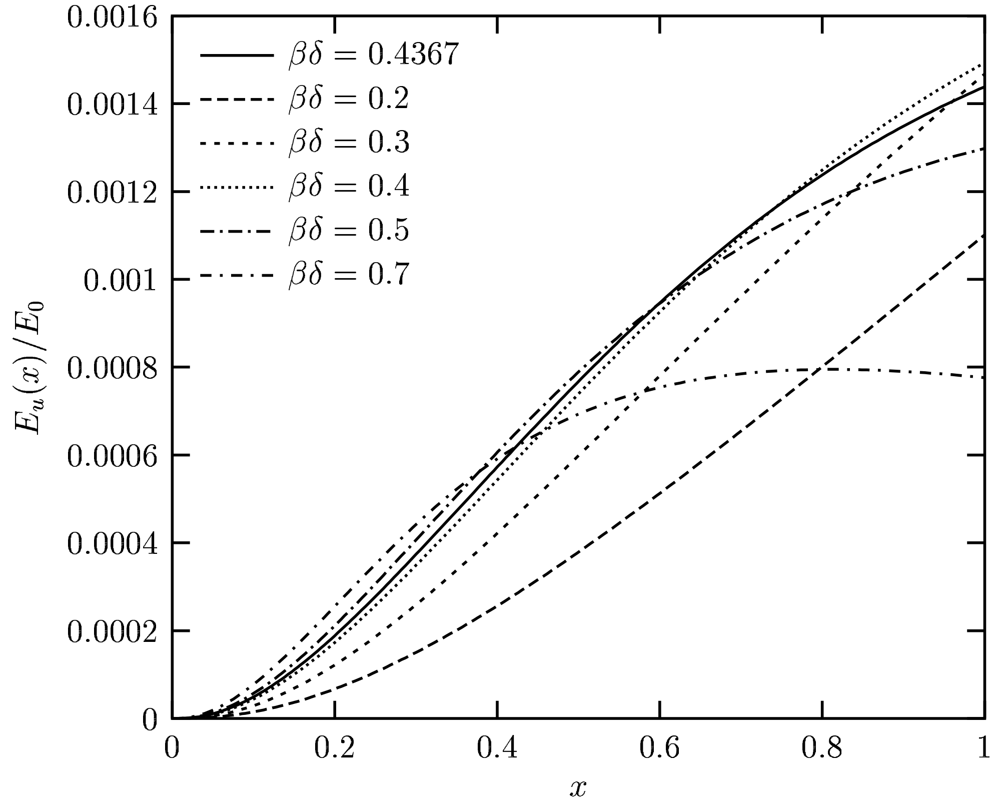
$\Rightarrow$ Mean flow distortion with respect to Blasius

Optimal perturbation – fixed E_0

Energy behavior $E_u(x)/E_0$ for varying $\beta\delta$ at fixed E_0



$E_0 = 1$

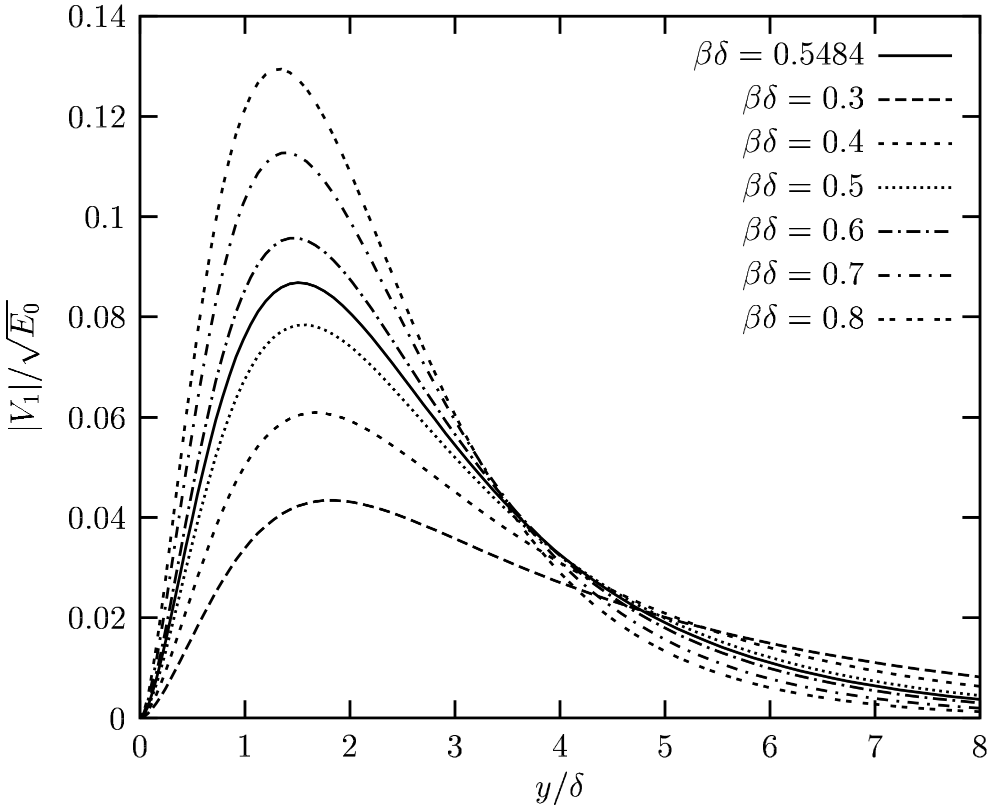


$E_0 = 1000$

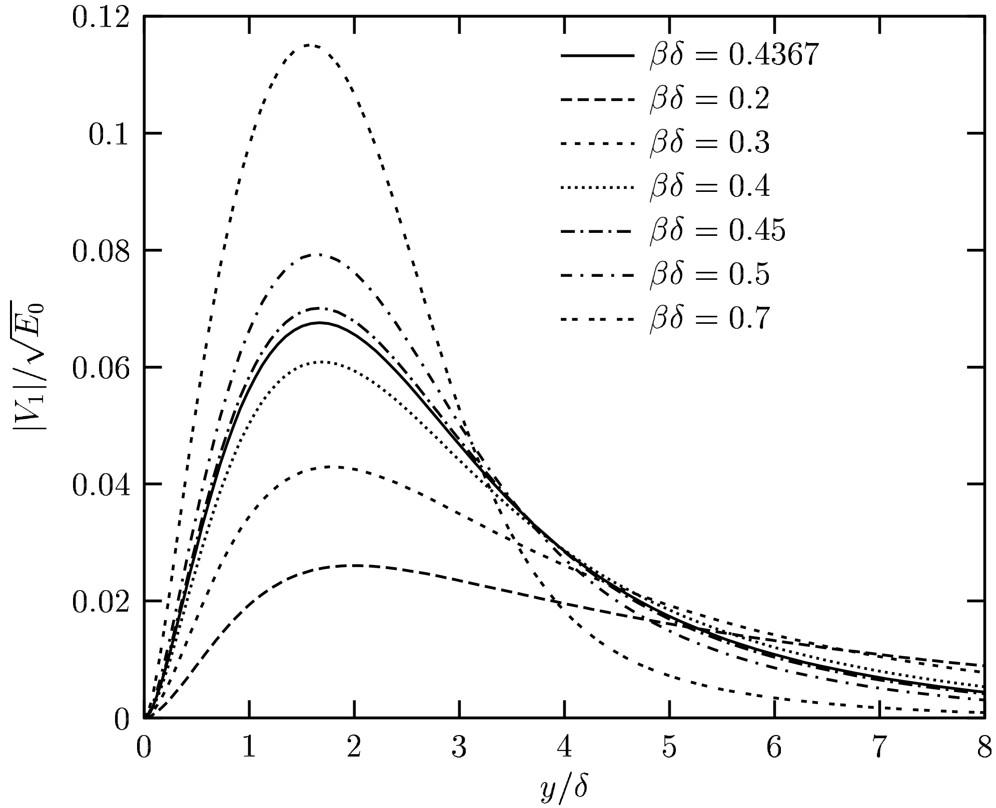
$\Rightarrow$ Saturation or effect of high $\beta\delta$?

Optimal perturbation – fixed E_0

Optimal perturbation $|V_1|/\sqrt{E_0}$ for varying $\beta\delta$ at fixed E_0



$E_0 = 1$

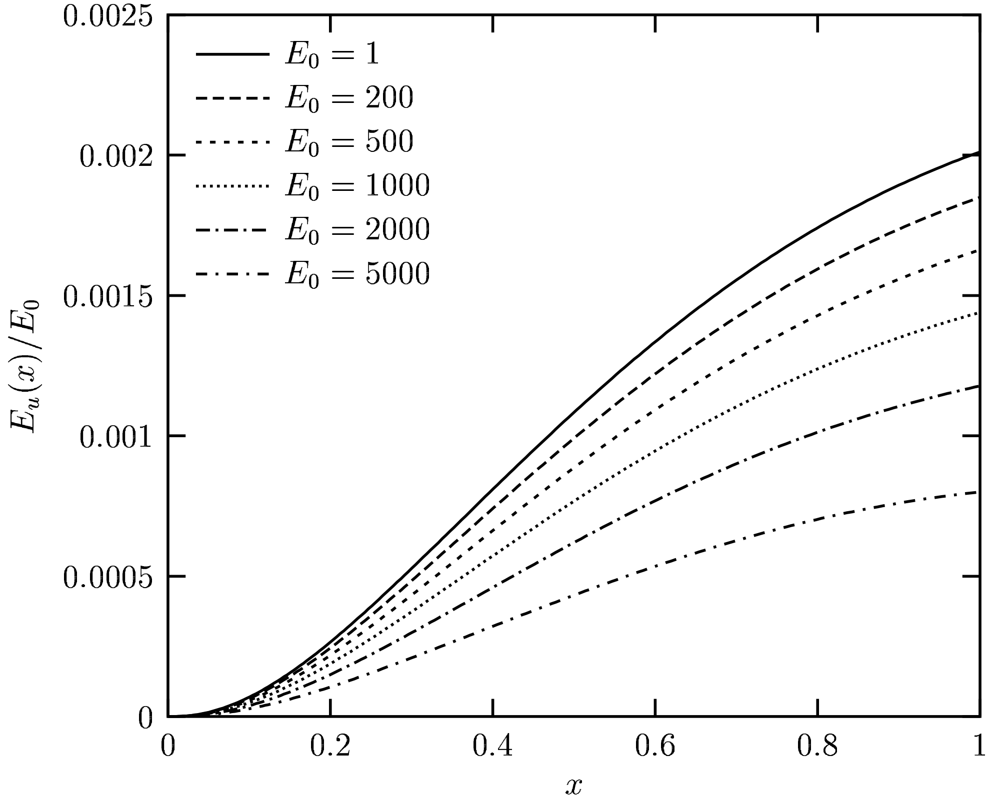


$E_0 = 1000$

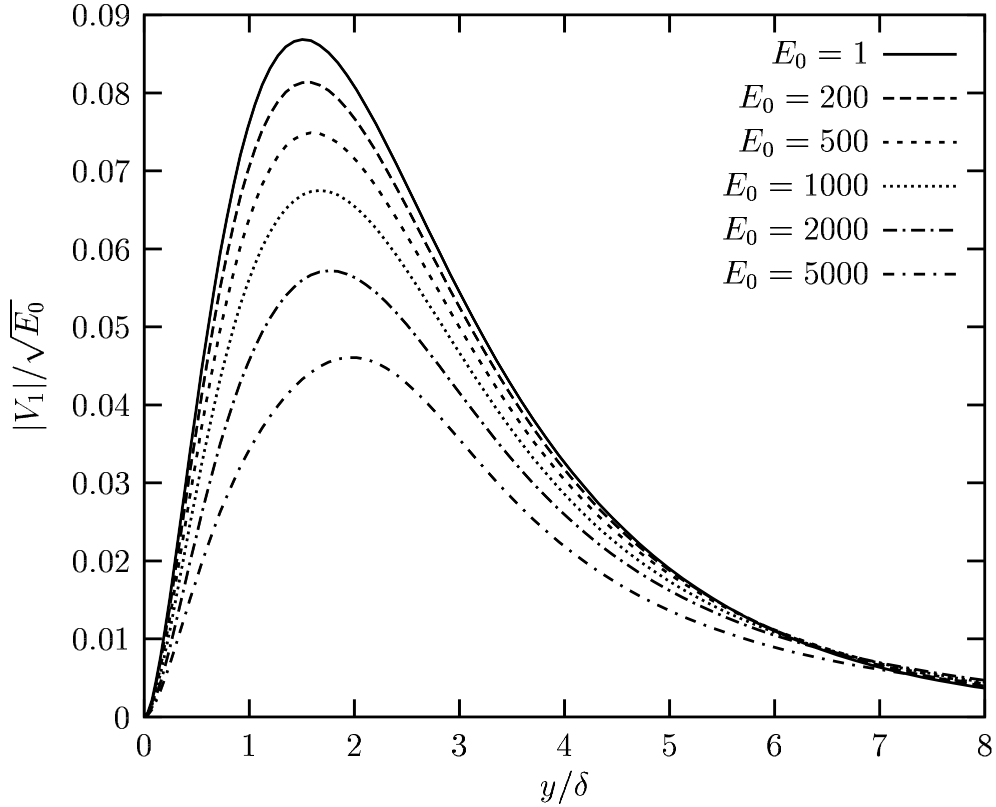
$\Rightarrow$ Effect of the wavenumber $\beta\delta$, weak dependence on E_0

Optimal perturbation – optimal $\beta\delta$

Comparisons at optimal $\beta\delta$ for different values of E_0 (initial energy)



Energy behavior $E_u(x)/E_0$

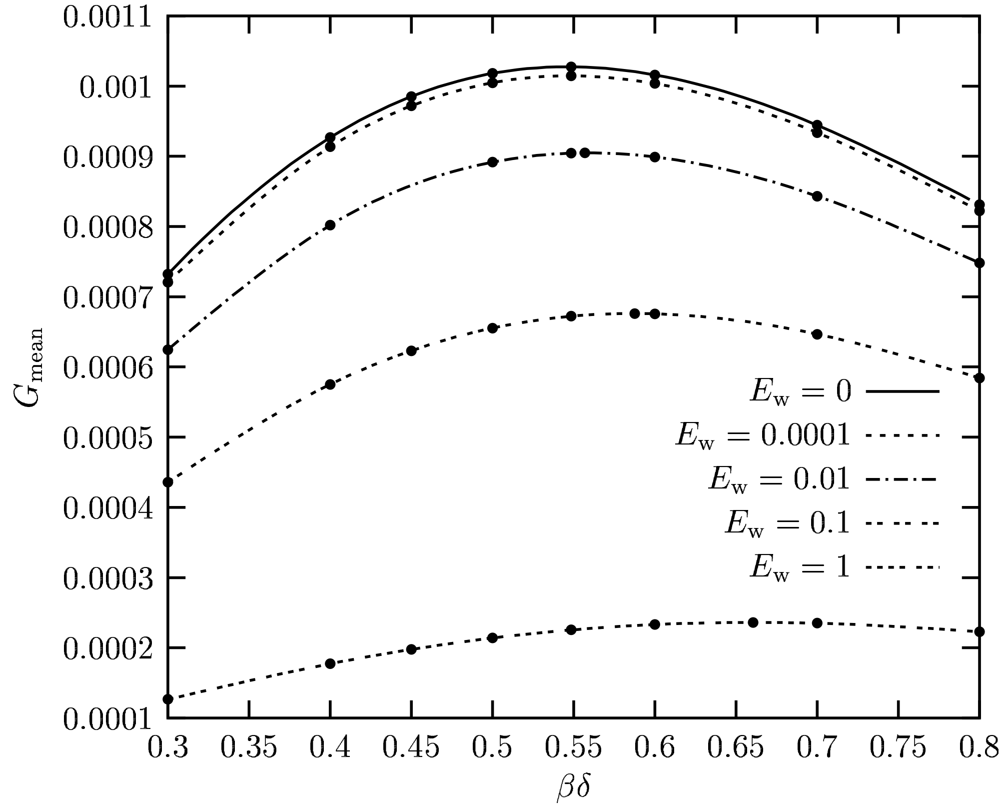


Optimal perturbation $|V_1|/\sqrt{E_0}$

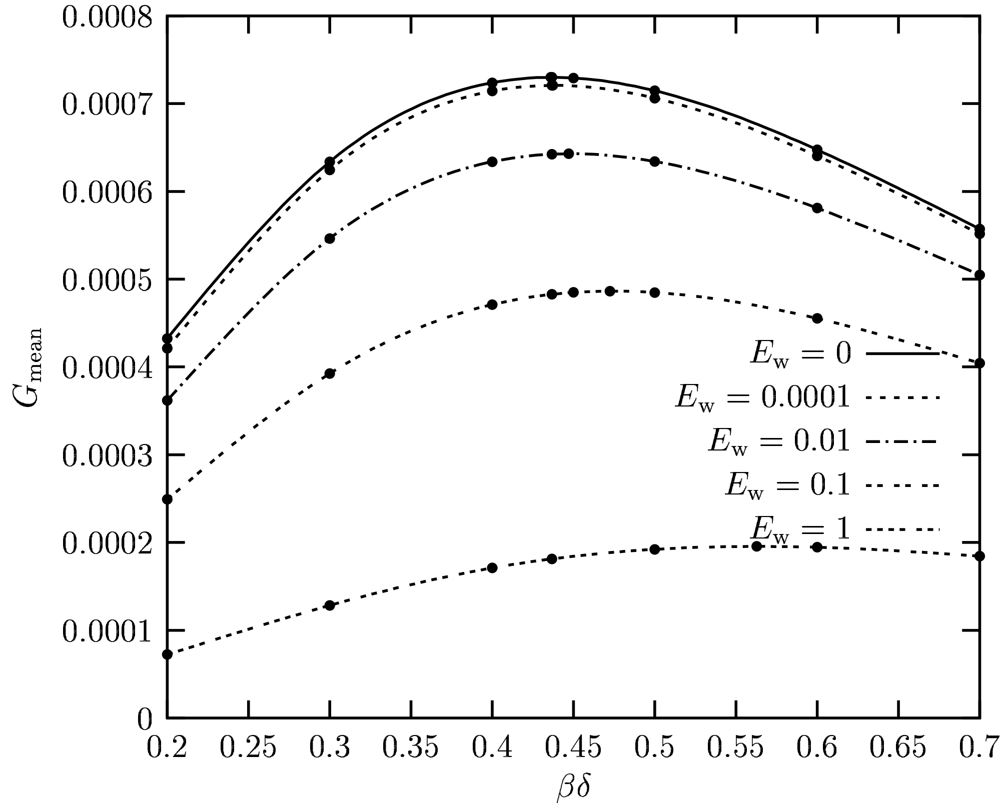
$\Rightarrow$ Much more regular behavior with varying E_0

Optimal control

Gain $G_{\text{mean}} = E_{\text{mean}}/E_0$ for varying control energy E_{w} and wavenumber $\beta\delta$



$E_0 = 1$

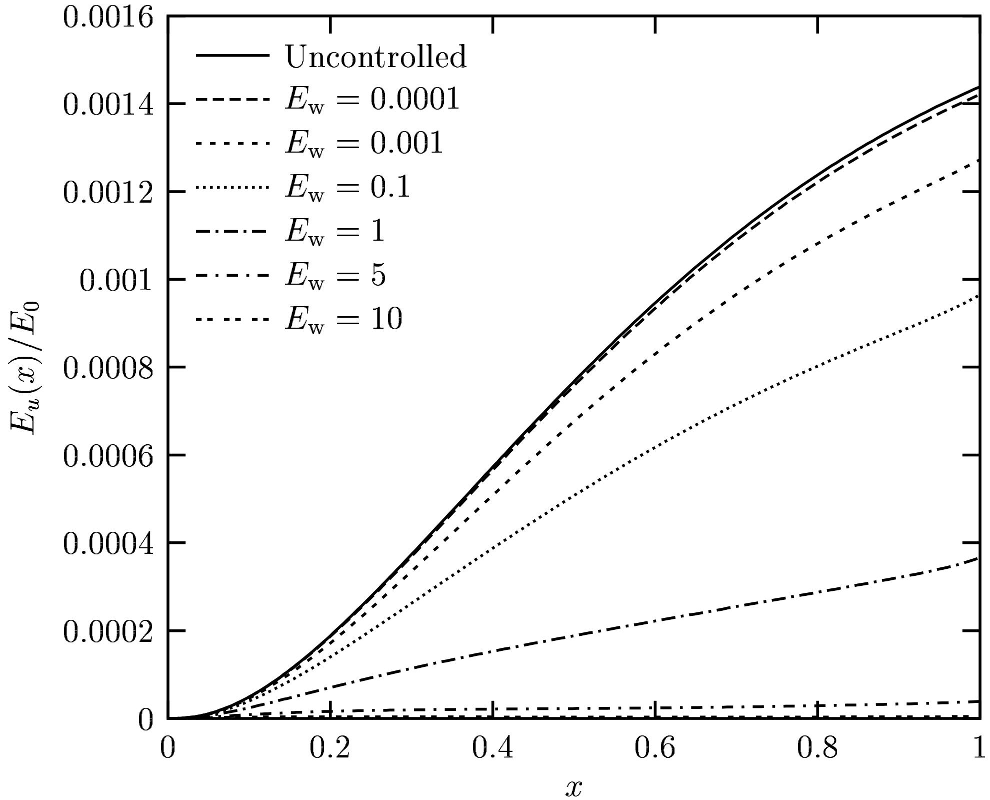


$E_0 = 1000$

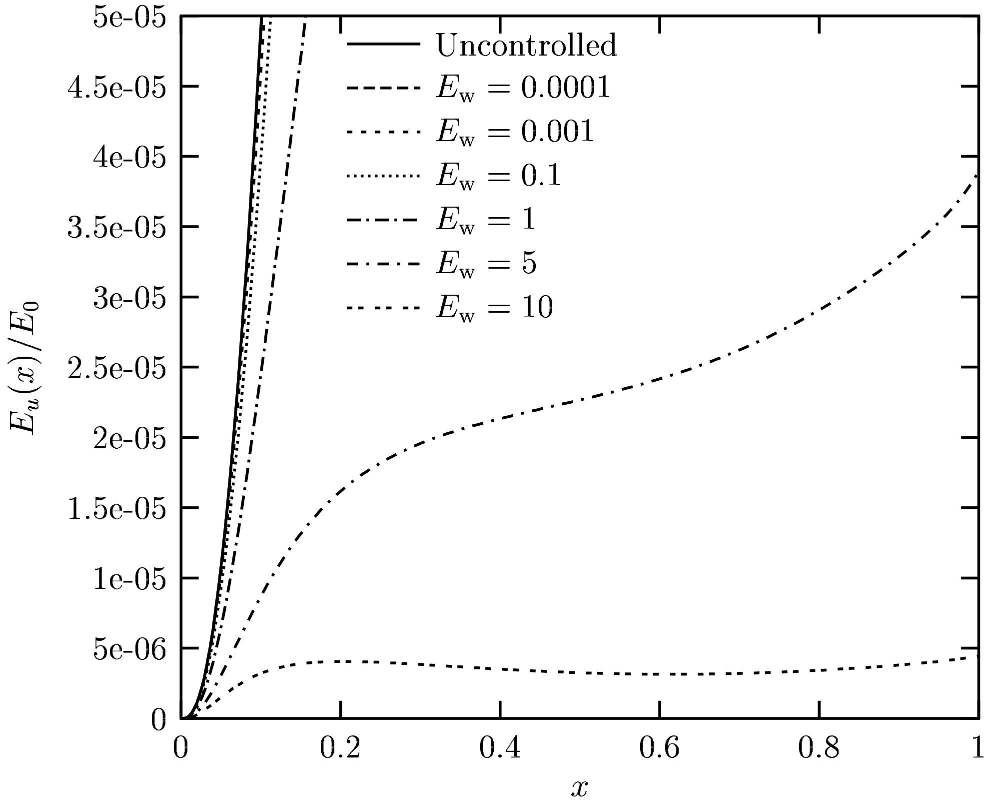
$\Rightarrow$ Weak dependence on the initial energy E_0

Optimal control – $E_0 = 1000$ $\beta\delta = 0.437$

Energy behavior $E_u(x)/E_0$ with varying control energy E_w at fixed $\beta\delta$



$E_u(x)/E_0$

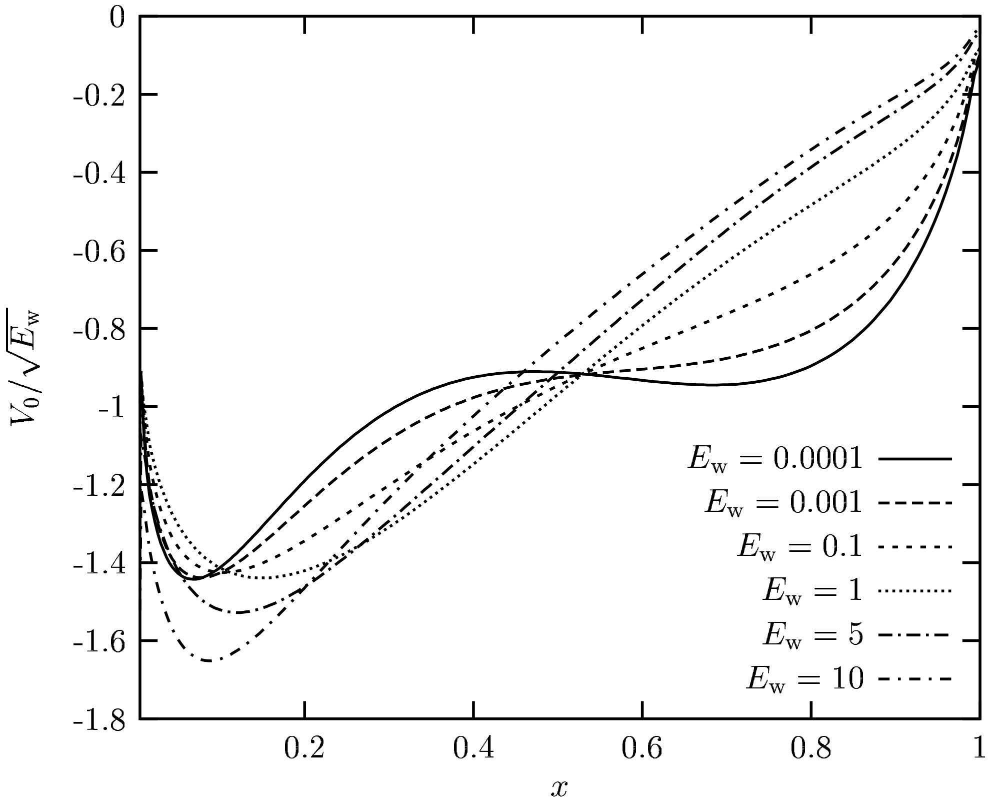


$E_u(x)/E_0$

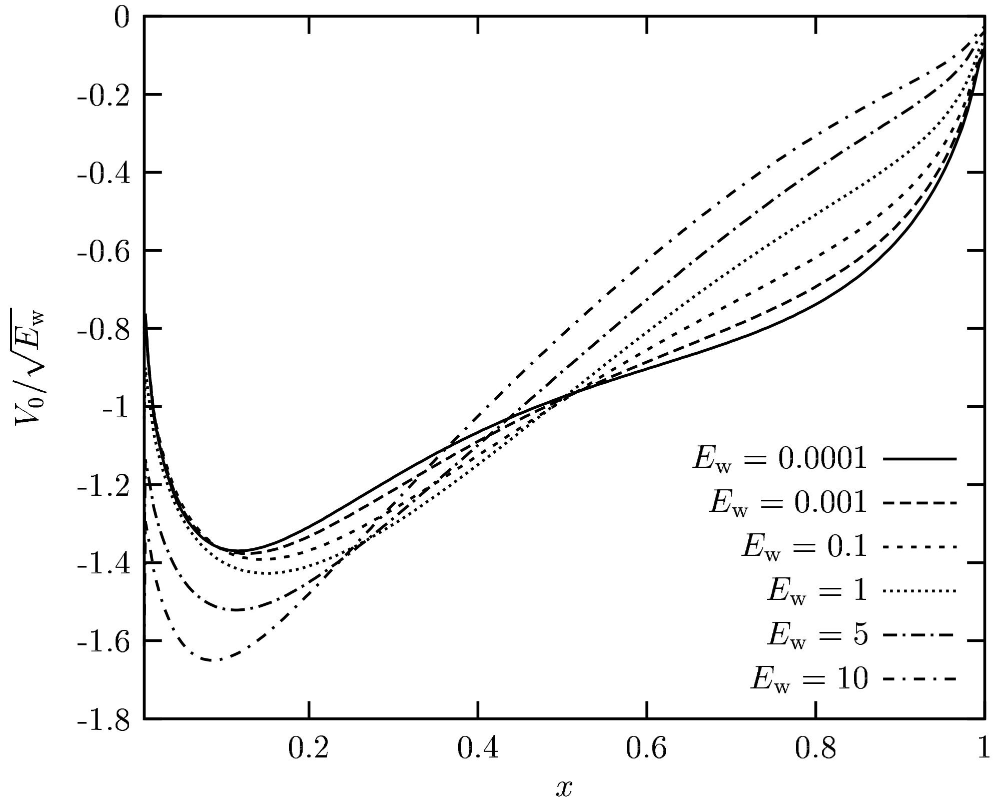
$\Rightarrow$ Remarkable attenuation of the perturbation energy for high values of E_w

Optimal control – fixed $\beta\delta$

Optimal control at the wall $V_0/\sqrt{E_w}$ for varying E_w at fixed $\beta\delta$



$E_0 = 1$ and $\beta\delta = 0.548$

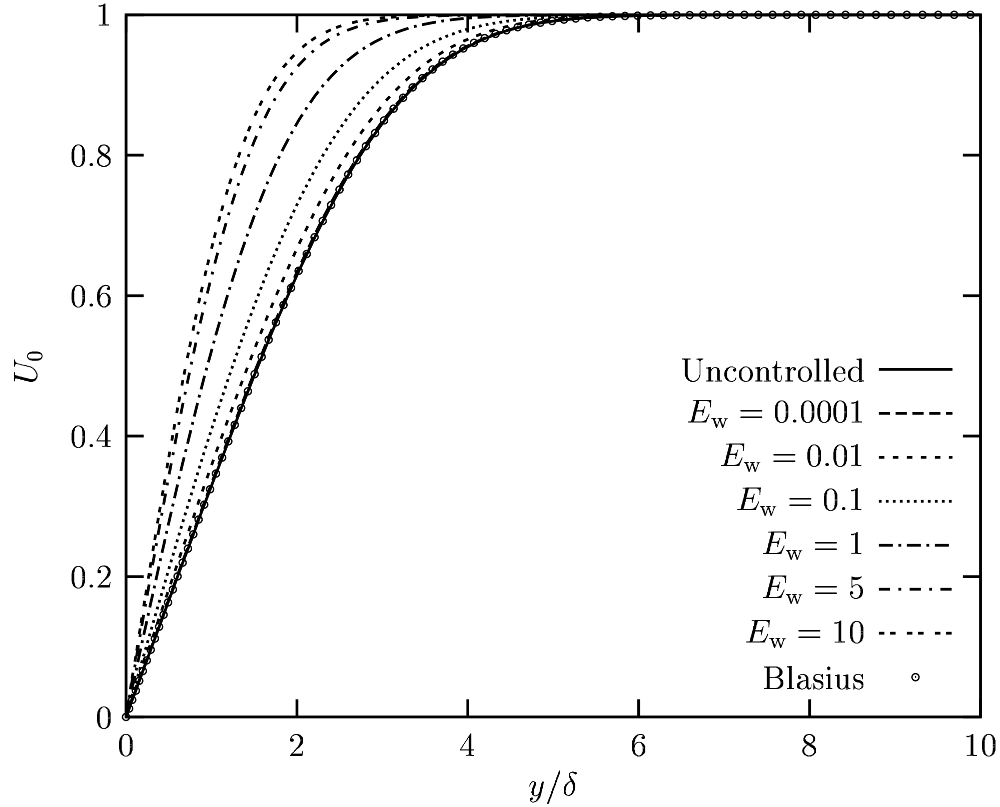


$E_0 = 1000$ and $\beta\delta = 0.437$

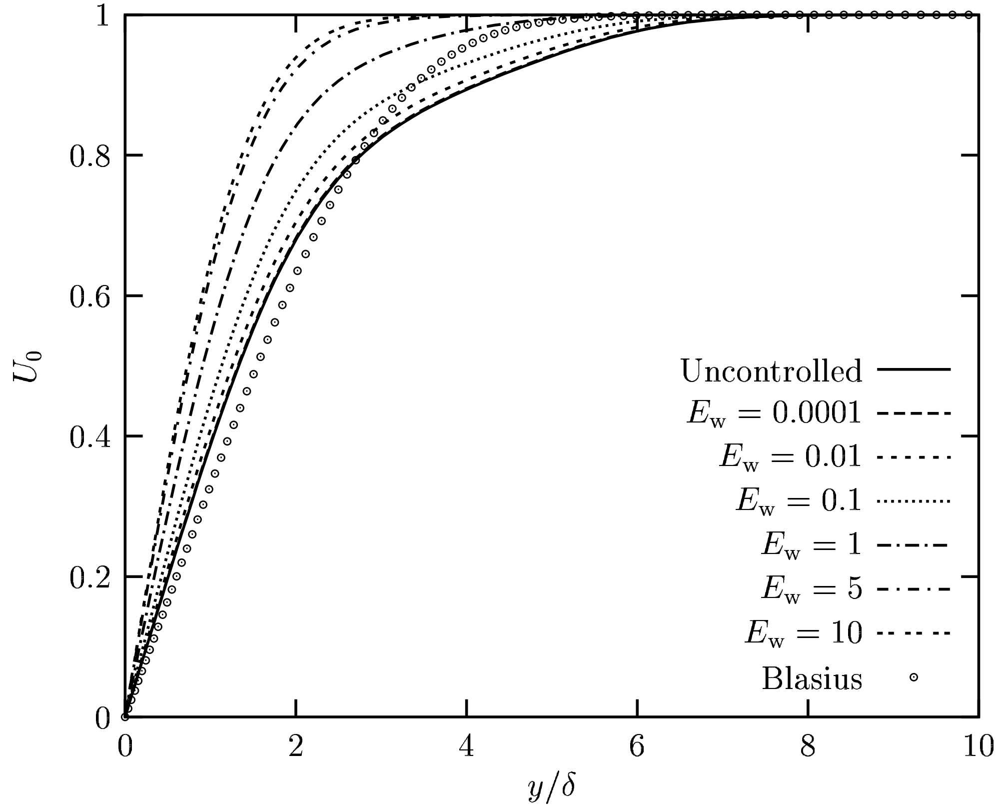
$\Rightarrow V_0 < 0$ (suction). Maximum control close to the LE. More regular profile at high E_w

Optimal control – fixed $\beta\delta$

Mean flow contribution (independent of z). Profiles of U_0 at the final station $x = 1$



$E_0 = 1$ and $\beta\delta = 0.548$

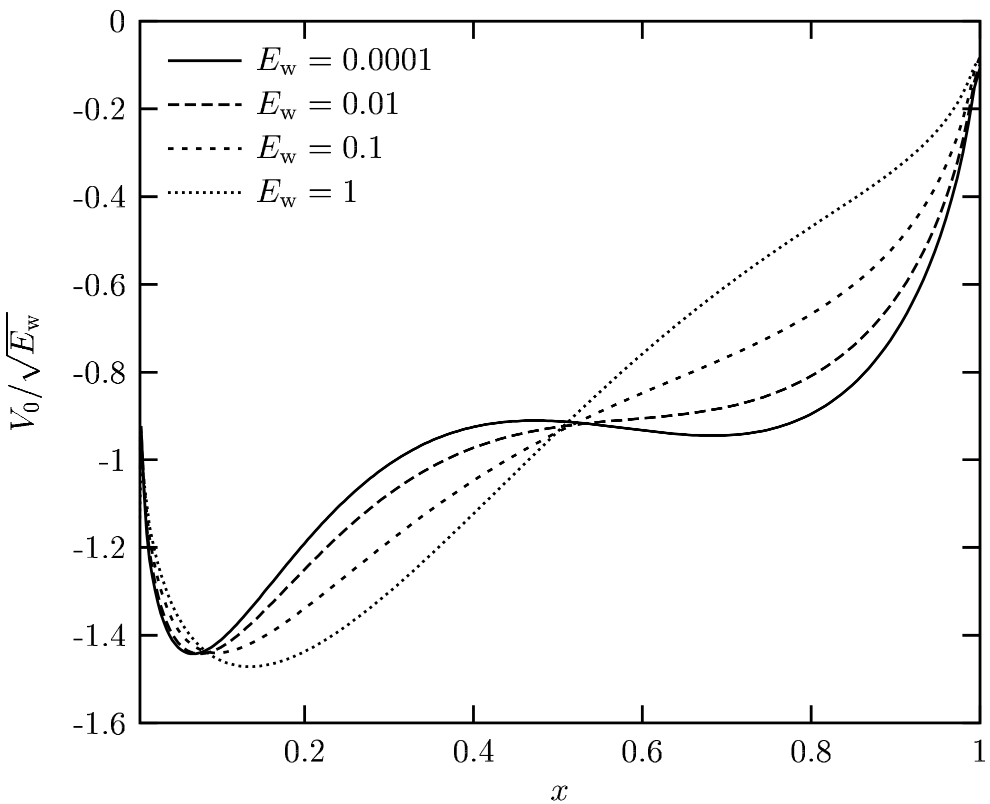


$E_0 = 1000$ and $\beta\delta = 0.437$

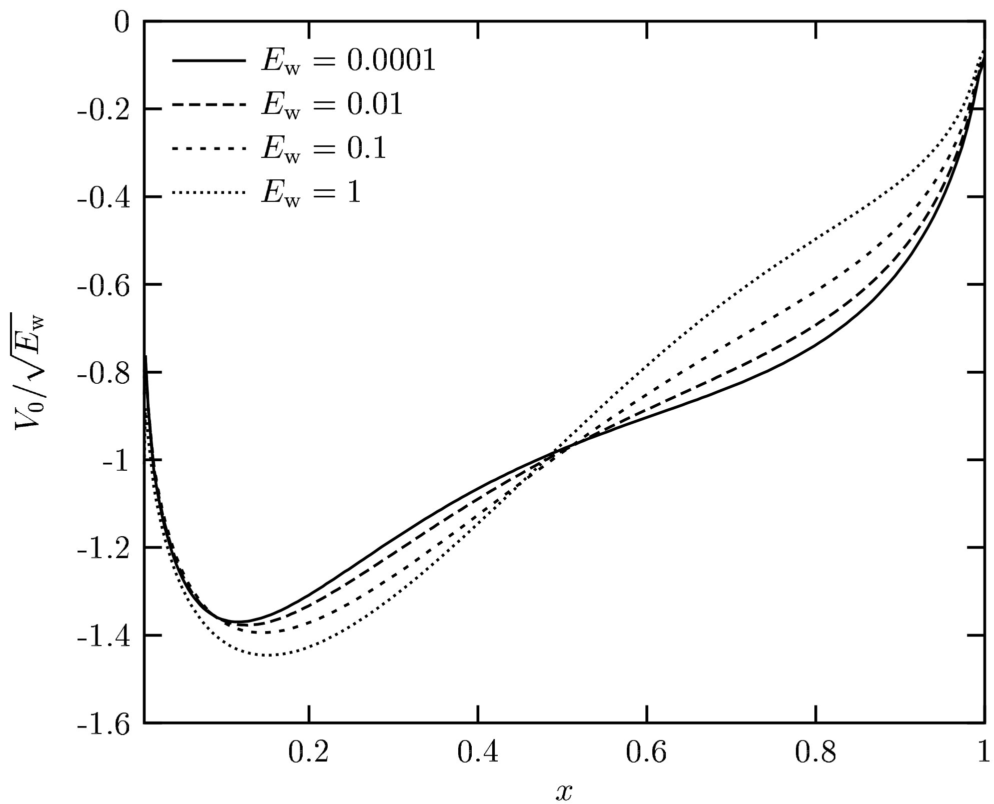
$\Rightarrow$ More regular profiles resembling accelerating Falkner–Skan ones

Optimal control – optimal $\beta\delta$

Optimal control at the wall $V_0/\sqrt{E_w}$ for varying E_w at optimal $\beta\delta$



$E_0 = 1$

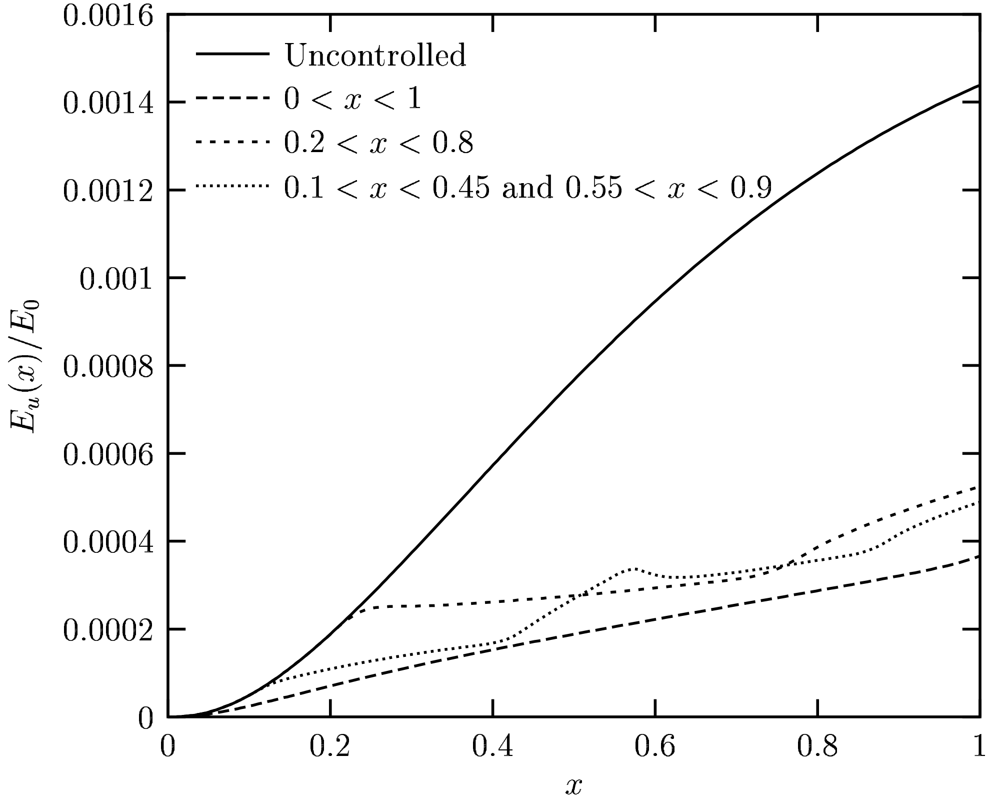


$E_0 = 1000$

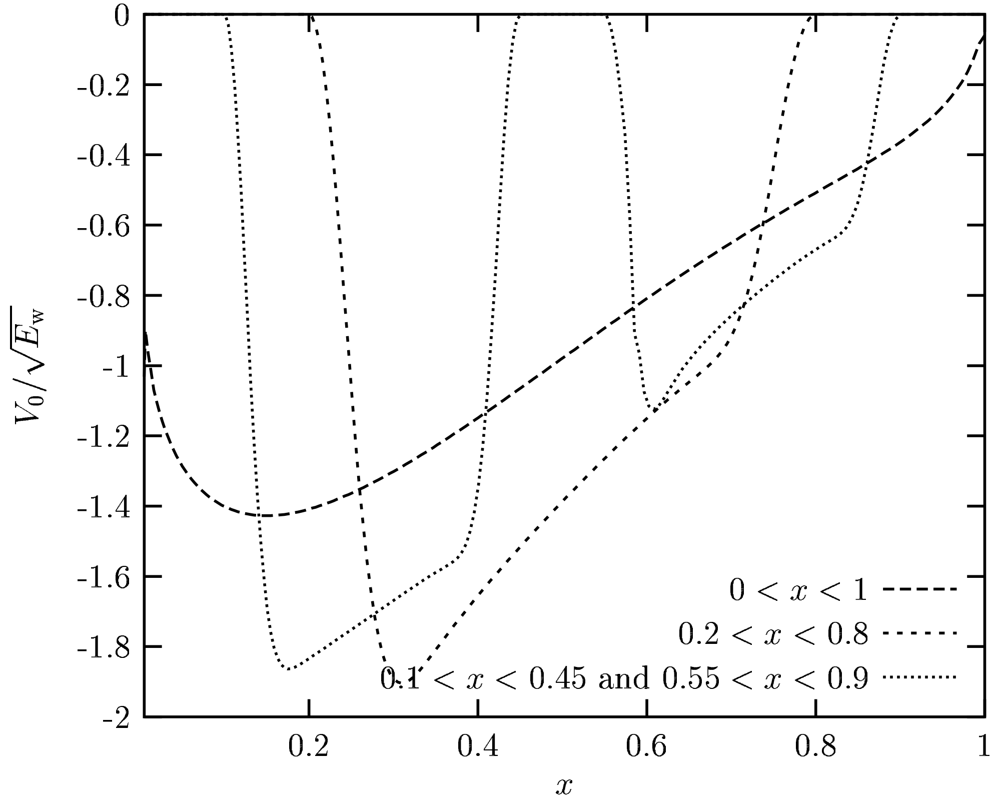
$\Rightarrow$ Effect of the initial energy E_0 on the regularity of the profile

Optimal control – finite window

Controlling of finite windows. $E_{\rm w} = 1$, $E_0 = 1000$ and $\beta\delta = 0.437$



Energy behavior $E_u(x)/E_0$

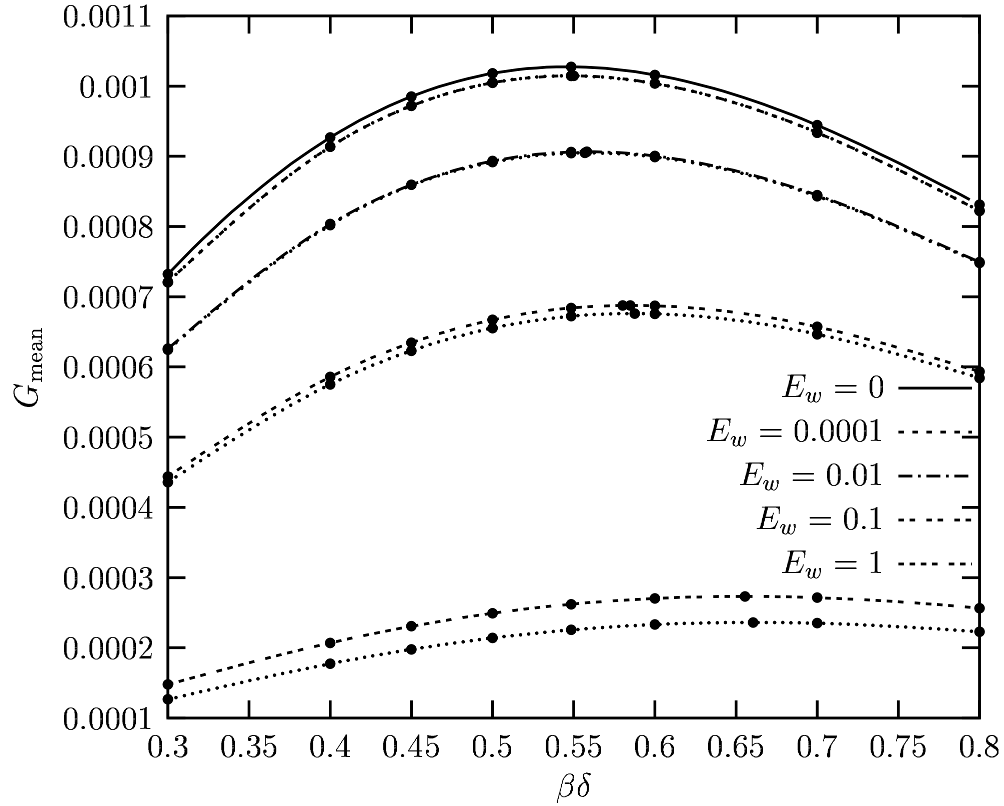


Optimal suction profile $V_0/\sqrt{E_{\rm w}}$

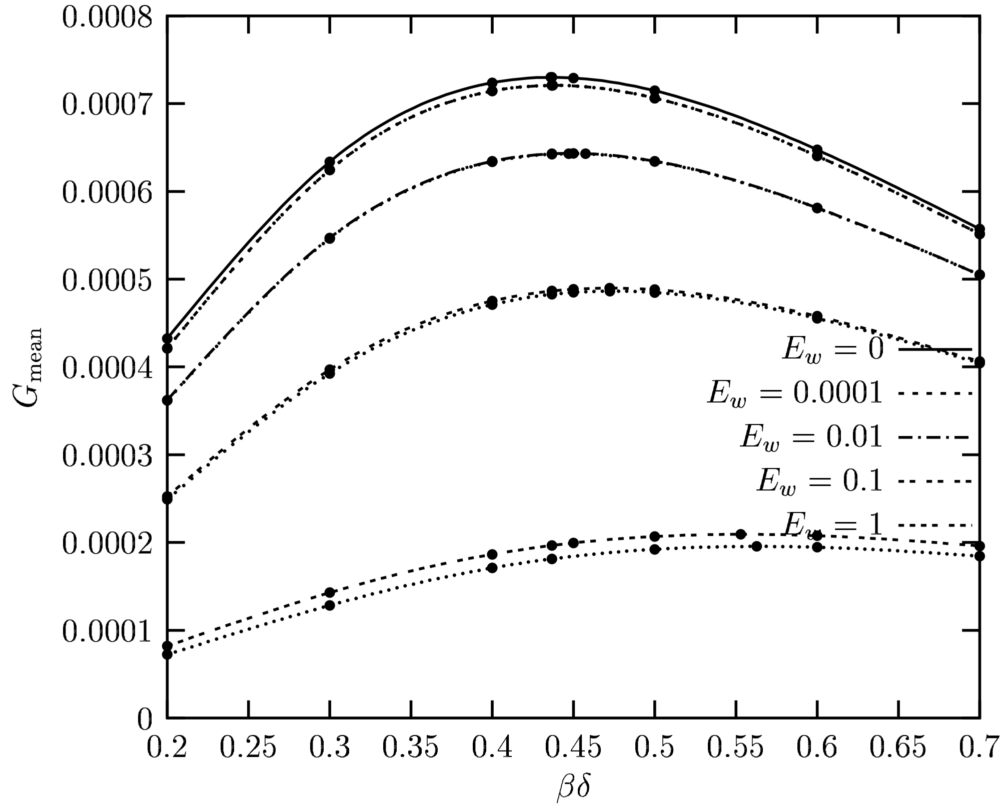
$\Rightarrow$ Controlling on multiple strips is less efficient

Robust control

Gain $G_{\text{mean}} = E_{\text{mean}}/E_0$ for varying control energy E_{w} and wavenumber $\beta\delta$



$E_0 = 1$

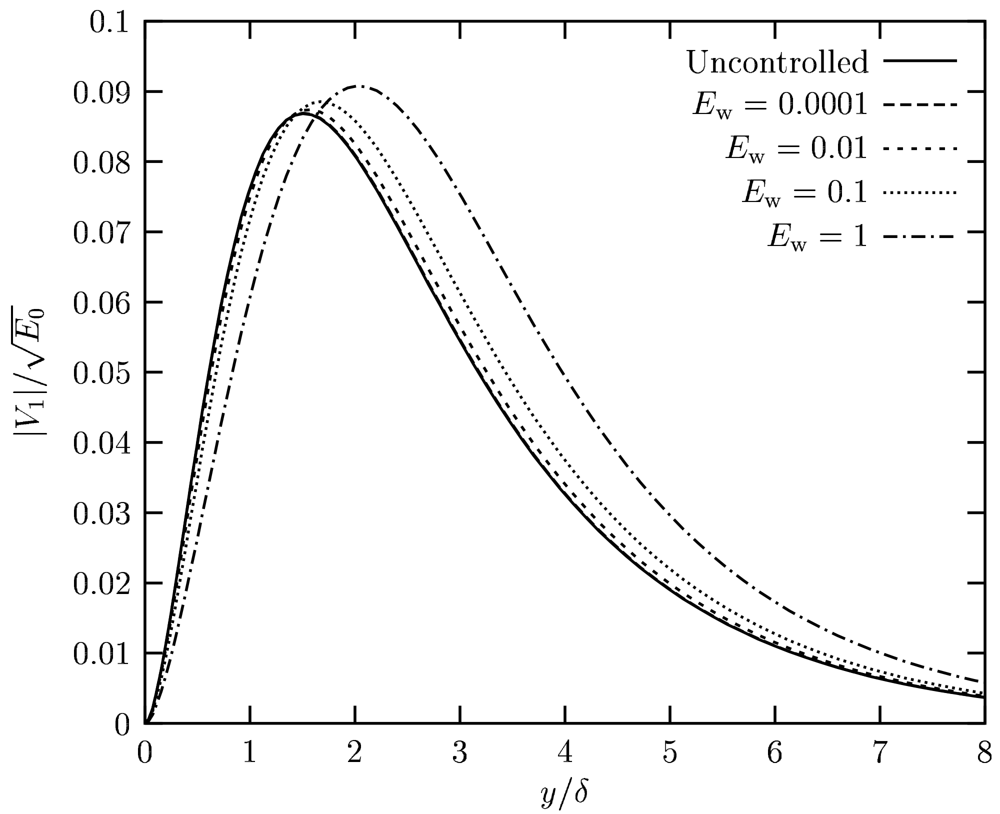


$E_0 = 1000$

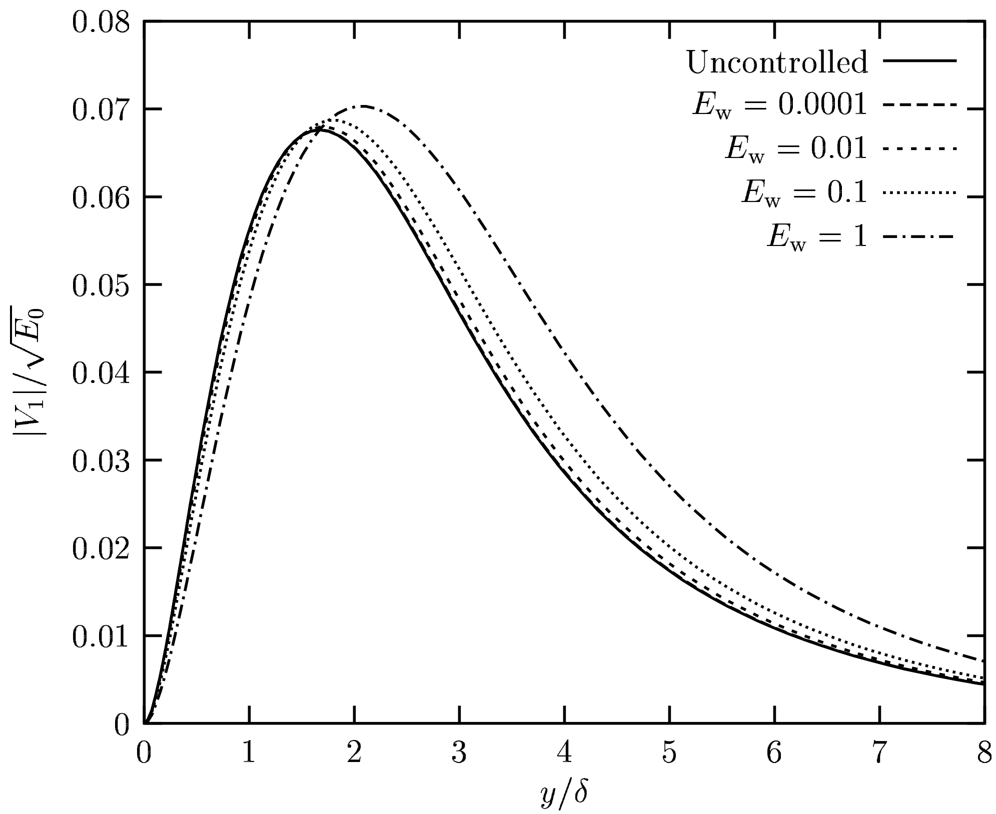
$\Rightarrow$ Robust control curves are always above optimal control curves.

Robust control – fixed $\beta\delta$

Optimal perturbation $|V_1|/\sqrt{E_0}$ for varying E_w at fixed $\beta\delta$



$E_0 = 1$ and $\beta\delta = 0.548$

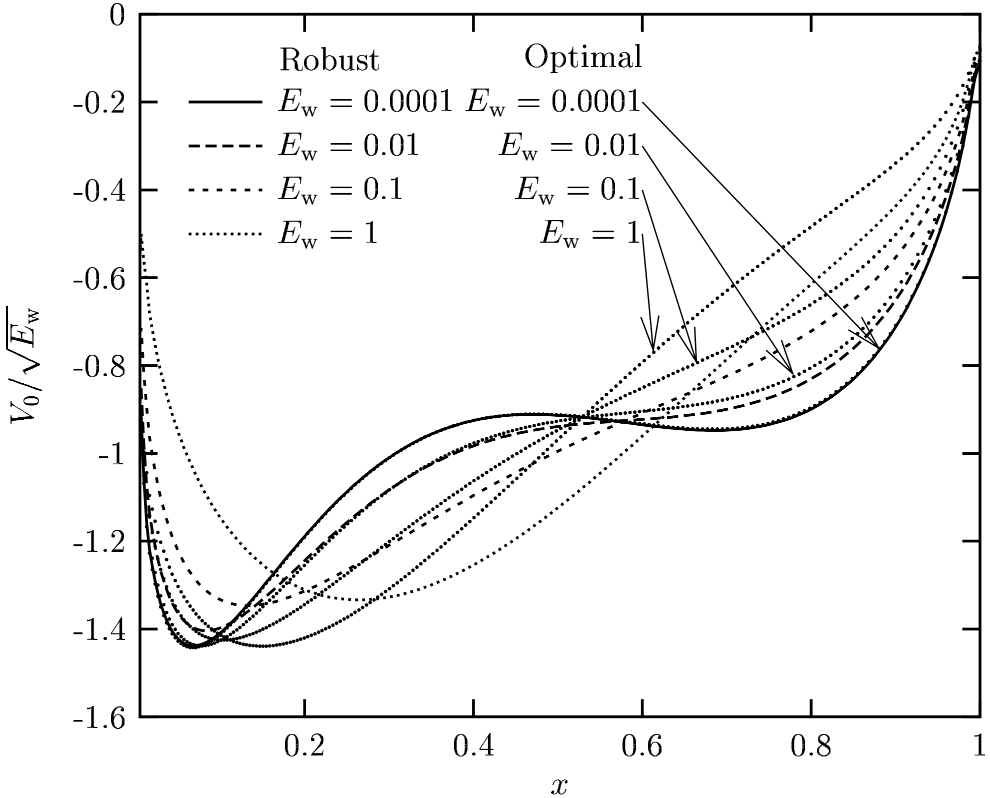


$E_0 = 1000$ and $\beta\delta = 0.437$

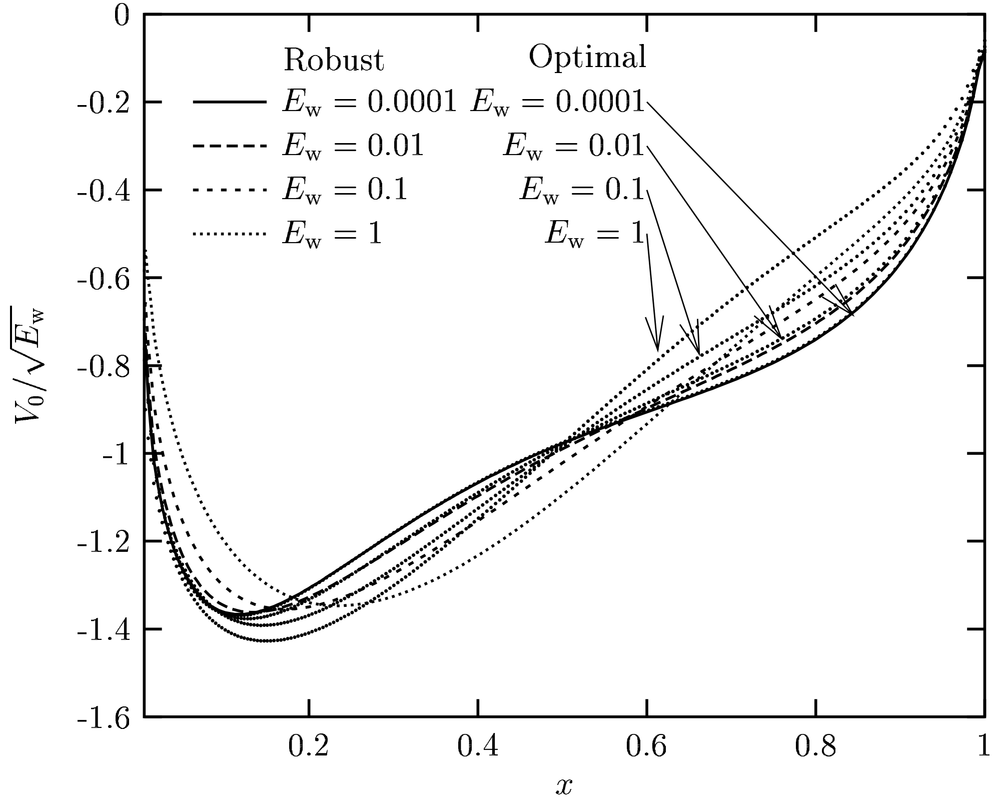
$\Rightarrow$ Shift of the maximum for increasing control energy E_w

Robust control – fixed $\beta\delta$

Optimal control at the wall $V_0/\sqrt{E_w}$ for varying E_w at fixed $\beta\delta$



$E_0 = 1$ and $\beta\delta = 0.548$

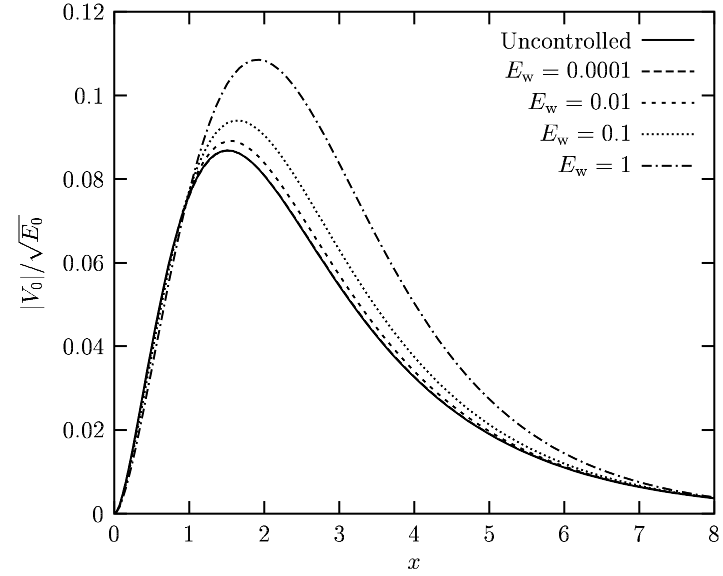


$E_0 = 1000$ and $\beta\delta = 0.437$

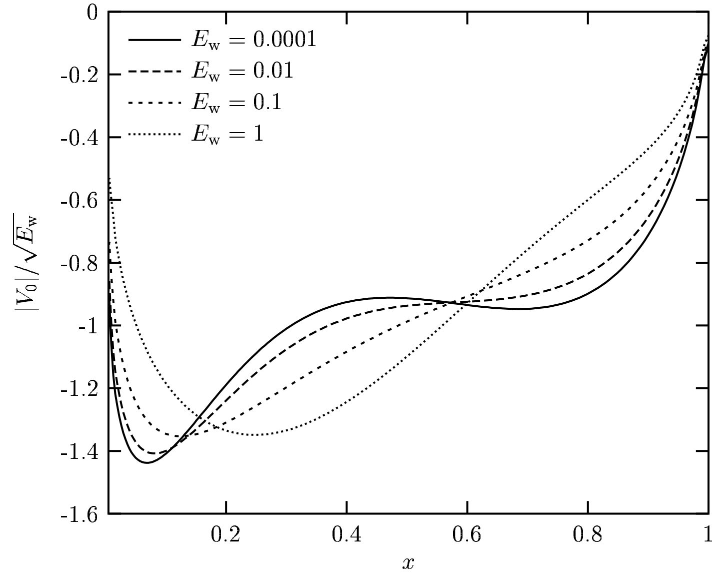
$\Rightarrow$ Stronger dependence of the optimal suction profile on E_w for low initial energy

Robust control – optimal $\beta\delta$

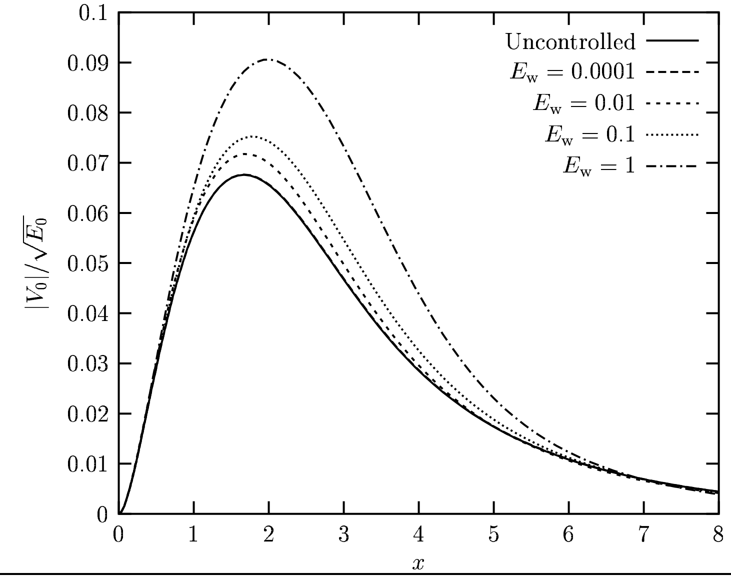
$E_0 = 1$



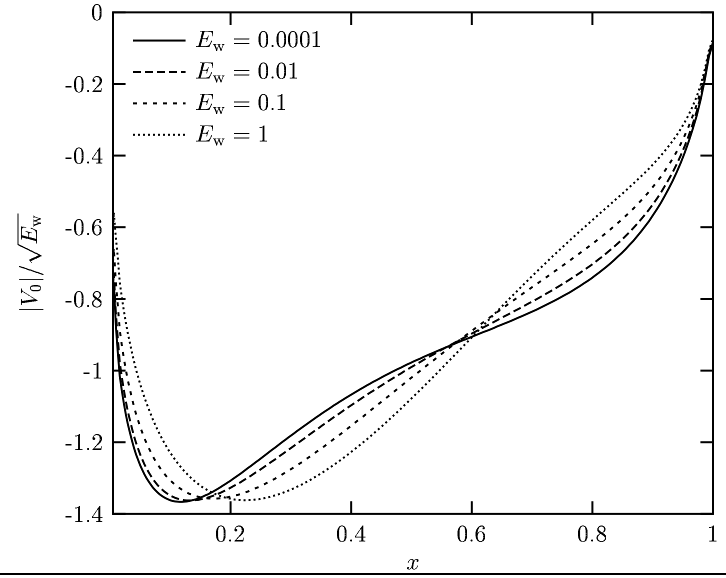
$E_0 = 1$



$E_0 = 1000$



$E_0 = 1000$

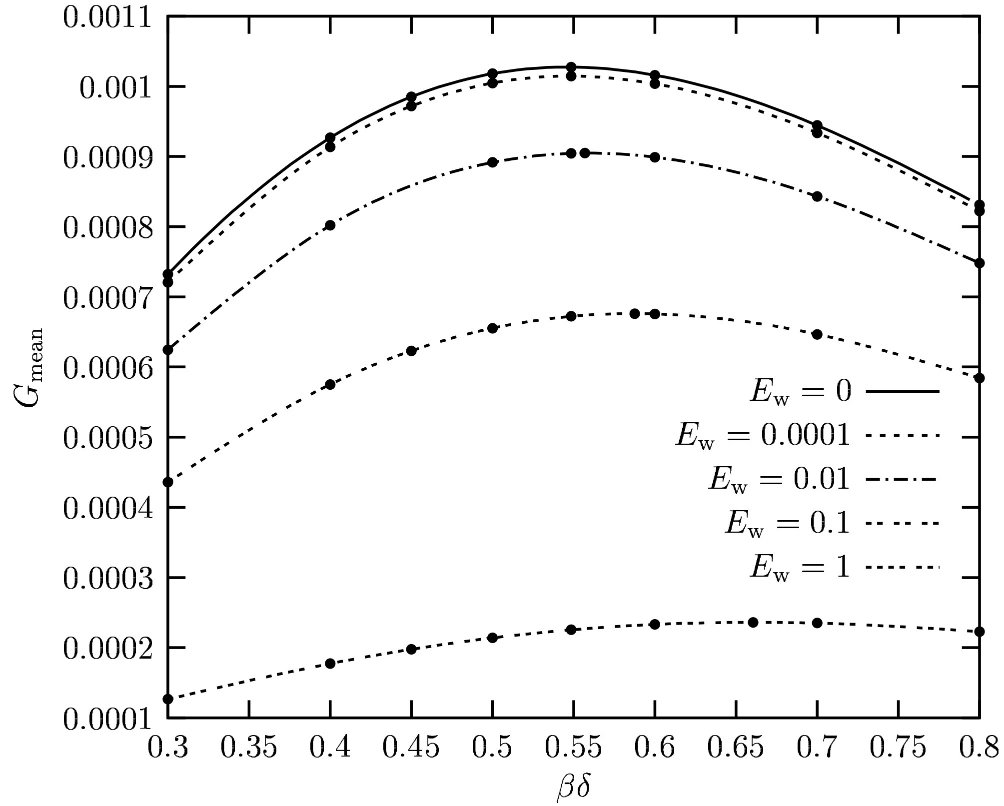


Conclusions

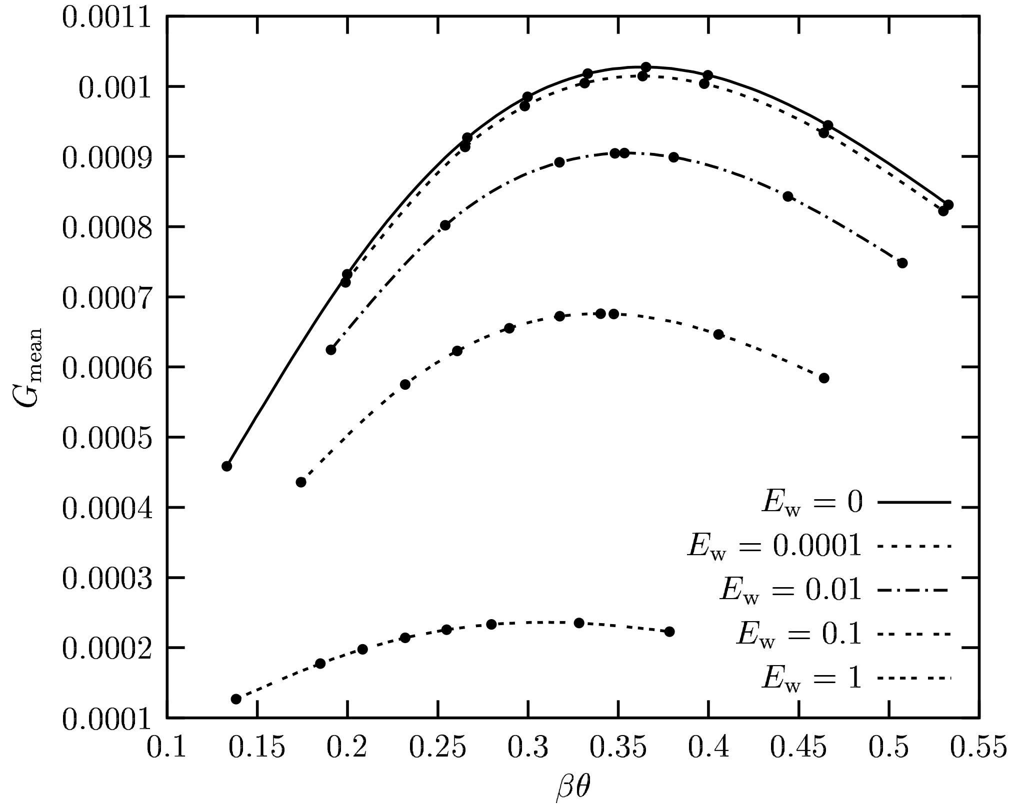
- $\surd$ **Three–dimensional**, incompressible, **nonlinear** boundary layer equations solved
- $\surd$ Optimization technique based on the (linear) **adjoint equations** of the direct (nonlinear) problem
- $\surd$ **Optimal perturbation**. In the linear case, previous results reproduced. In the nonlinear case, extended study for varying E_0 and $\beta\delta$. With increasing E_0, optimal $\beta\delta$ moves. Distortion effects on the unperturbed flow (mode zero – independent of z)
- $\surd$ **Optimal control**. Comparisons for varying initial energy, control energy and wavenumber. Maximum control always located close to the leading edge. Unperturbed flow profiles resemble accelerating Falkner–Skan ones. Controlling on multiple strips is not convenient
- $\surd$ **Robust control**. Robust control curves are always above optimal control ones. Greater difference with increasing control energy

Optimal control

Gain $G_{\text{mean}} = E_{\text{mean}}/E_0$ for $E_0 = 1$ and varying wavenumber β



$$\delta = L/\sqrt{Re_L} = \sqrt{\nu L/U_\infty}$$



$$\theta = \int_0^{+\infty} \frac{U_0}{U_\infty}\left(1 - \frac{U_0}{U_\infty}\right) dy$$